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Superstring in the plane-wave background with RR flux as a conformal field theory

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Abstract: We study the type IIB superstring in the plane-wave background with Ramond-Ramond flux and formulate it as an exact conformal field theory in operator formalism. One of the characteristic features of the superstring in a consistent background with RR flux, such as the $AdS_5 \times S^5$ and its plane-wave limit, is that the left- and the right-moving degrees of freedom on the worldsheet are inherently coupled. In the plane-wave case, it is manifested in the well-known fact that the Green-Schwarz formulation of the theory reduces to that of free massive bosons and fermions in the light-cone gauge. This raises the obvious question as to how this feature is reconciled with the underlying conformal symmetry of the string theory. By adopting the semi-light-cone conformal gauge, we will show that, despite the existence of such non-linear left-right couplings, one can construct two independent sets of quantum Virasoro operators in terms of fields obeying the free-field commutation relations. Furthermore, we demonstrate that the BRST cohomology analysis reproduces the physical spectrum obtained in the light-cone gauge.

Keywords: BRST Quantization, AdS-CFT Correspondence, Penrose limit and pp-wave background, Conformal Field Models in String Theory.

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Contents

1. Introduction and summary

Undoubtedly, the AdS/CFT correspondence [1 – 3] is one of the most profound structures in string theory. In the past 10 years, an impressive collection of evidences have been accumulated in favor of this remarkable conjecture. On the CFT side, fairly detailed analyses have recently become possible, in particular for the $N = 4$ super-Yang-Mills theory, with the powerful assumption of integrability [4 – 7] as well as with the state of the art perturbative techniques [8 – 10]. On the other hand, the corresponding developments on the AdS side

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have been available mostly in the classical or semi-classical regimes [11 – 14]. Understanding of the stringy aspects has been comparatively slow due to the difficulty of solving the string theory in the relevant curved backgrounds with large Ramond-Ramond(RR) flux. In any case, the fundamental mechanism of this correspondence is yet to be unravelled.

There are strong reasons to believe that the presence of the RR flux must play a key role in this strong-weak duality. Most directly, the basic relation $4\pi g_s N = g_{\rm YM}^2 N = R^4/\alpha'^2$ expresses the balance between the curvature and the RR flux, which keeps the $AdS_5 \times S^5$ type curved geometry from gravitationally collapsing. Unfortunately, however, the treatment of the RR flux (together with the curved background) is precisely the major obstacle which has been hampering the progress of the string side of the AdS/CFT correspondence.

One notable exception is the superstring in the plane-wave limit of $AdS_5 \times S^5$ [15 – 17]. As was first shown in [18], by adopting the light-cone gauge in the Green-Schwarz formalism [19 – 21], the worldsheet theory in this case becomes simply a collection of free massive bosons and fermions. This fact was exploited in [22] to initiate a detailed comparison of the spectrum of the energy of the string and that of the anomalous dimensions of the corresponding gauge-invariant operators in super-Yang-Mills theory. Subsequent surge of researches on this so-called BMN limit advanced our understanding of the nature of the AdS/CFT correspondence enormously (see [23, 24] for reviews).

However, a number of important aspects of the string theory in this background are still to be clarified. For one thing, the analysis of the interactions of the string is not straightforward. Although the three point vertex has been computed using the light-cone string field theory [25 – 34], higher point functions have not yet been constructed. Another important aspect which needs to be better understood is the modular property. Because the modular S-transformation affects the light-cone gauge condition itself, the partition function is not modular invariant but only modular "covariant" (in a certain sense) [35, 36]. Including these questions, one should understand the string theory in this background more fully as a precious prototypical model, in particular for understanding the role of the large RR flux in the AdS/CFT correspondence.

One major factor which has been preventing further understanding of the theory is the lack of conformal invariance in the light-cone gauge formulation. As a string theory, it should be possible to formulate it as an exact conformal field theory (CFT) and make use of its tight structures and poweful techniques.[1] Once it is achieved in a tractable manner, we should be able to construct the interaction vertices more easily and discuss the modular invariance in a proper setting. Also, it should serve as a basis for constructing an operator version of the covariant pure spinor formalism in this background [42 – 44].

One would expect, however, that such a CFT formulation is not so straightforward. This is because the theory at hand has quite an unusual feature. Namely, just as in the $AdS_5 \times S^5$ background, the left- and the right-moving degrees of freedom on the worldsheet are coupled in the plane wave background right from the beginning. In the light-cone gauge, this is manifested as the massive nature of the bosons and the fermions. On the

[1] For the CFT formulation of a superstring in a class of AdS_3 backgrounds with RR flux, there have been a large number of investigations. See for example [37 – 41].

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other hand, a CFT description means, by definition, that the “left” and the “right” sectors must “decouple” in the sense that they form representations of two mutually commuting Virasoro algebras. It is extremely interesting to see how these two features are reconciled. Another obvious difficulty is that in the conformal gauge the action is no longer quadratic and even the classical analysis, let alone the quantum extension, would become quite non-trivial.

In this paper, we shall show that, despite such anticipated obstacles, it is possible to achieve an exact CFT description, not only classically but quantum-mechanically as well, in an operator formalism. In the case with the NSNS flux, a similar CFT formulation was recently achieved in the RNS formalism using the canonical quantization method [45, 46]. As we shall see, with the RR flux the left-right coupling is much more inherent and such a canonical method encounters a severe difficulty. We will overcome this difficulty by the phase-space formulation, which does not require the knowledge of the solutions of the equations of motion.

Let us now give a summary of our results, which at the same time serves to indicate the organization of the paper.

We begin the analysis at the classical level in section 2. We take our basic Lagrangian to be the one constructed by Metsaev [18] in the Green-Schwarz formalism in the semi-light-cone(SLC) conformal gauge [47 − 49]. As briefly reviewed in section 2.1, it is composed of the string coordinates $X^\mu = (X^+, .X^-, X^I), (I = 1 \sim 8)$ and the two sets of 16-component Majorana spinors $\theta^A_\alpha, (A = 1, 2)$ and contains quartic interactions expressing the coupling to the curved space and to the RR flux through a “mass” parameter μ.

For a pedagogical reason, we will first describe, in section 2.2, what happens if one tries to treat this system by the canonical method. Despite non-linearity, the equations of motion can be solved and the general solutions for all the basic fields are obtained. However, at this stage one already encounters a sign of difficulty. Except for X^+, which is a free field, all the other fields are expanded in terms of the basis functions, to be called u_n and $\tilde{u}_n$, which not only depend on the modes of X^+ but also on the worldsheet light-cone coordinates $\sigma_\pm = t \pm \sigma$ *inseparably* due to the presence of μ.

To see how this is compatible with conformal invariance, we compute the energy momentum tensors $\mathcal{T}_\pm (= T_{\pm\pm})$ and substitute the solutions of the equations of motion. We then find that indeed $\mathcal{T}_\pm$ become functions of $\sigma_\pm$ respectively but in a peculiar manner. All the dependence on X^I, X^- and θ^A_α disappear and $\mathcal{T}_\pm$ collapse to exceedingly simple expressions involving X^+ and unknown “holomorphic functions” $f_\pm(\sigma_\pm)$, which appear in the solution of X^-. This situation is not smoothly connected to the flat case with $\mu = 0$. These functions are to be determined by the requirement that the basic fields and their conjugate momenta must satisfy the canononical Poisson bracket relations. The problem is that, although the Poisson brackets can be defined in the usual manner, it is practically impossible to analyze what $f_\pm$ should be, due to the highly complicated completeness relations for the operator-valued basis functions u_n and $\tilde{u}_n$. In this sense the canonical analysis “fails” even at the classical level.

To overcome this difficulty, we turn, in section 2.3, to the phase-space formulation. Although the method itself is completely standard, a crucial observation is that for theories, such as a string theory, where the Hamiltonian is a member of a large symmetry algebra, the dynamics can be encoded in the "kinematics", namely its representation theory. This allows us to focus on the Virasoro algebra structure *at one time slice*, say at $t = 0$, without recourse to the knowledge of the equations of motion. In spite of the presence of left-right couplings, one can indeed verify that $\mathcal{T}_\pm$ satisfy two mutually commuting sets of Virasoro algebras at the classical level.

We then turn, in section 3, to the quantum analysis. As we employ the phase-space formulation, the quantization of the basic fields, described in section 3.1, is straightforward. A great advantage of our formulation is that although these quantized fields time-develop non-trivially according to the full non-quadratic Hamiltonian, they obey the free-field commutation relations. What is non-trivial, however, is to find the appropriate normal-ordering prescription for the quantum Virasoro operators. In section 3.1.1, we define what we will call the phase-space normal-ordering and compute the commutators of the Virasoro operators. We find that, with an addition of a suitable quantum correction, they form two independent sets of Virasoro algebas with central charge 26, as desired. We also examine, in section 3.1.2, another scheme, to be called massless normal-ordering, which appears more natural for $\mu = 0$. Extending it to the $\mu \neq 0$ case, we find that the Virasoro commutators produce operator anomalies proportional to μ^2, which cannot be removed by quantum corrections. Some details of these computations are displayed in appendix A.

Having found the quantum Virasoro operators, it is straightforward to construct the BRST operator and study its cohomology. This is performed in section 4.1. Just as in the usual free string theory, the physical states will be identified as those in the transverse Hilbert space $\mathcal{H}_T$, i.e. without the non-zero modes of $X^\pm$, their conjugates and the ghosts, satisfying the Hamiltonian and the momentum constraints $H = P = 0$. These constraints can be easily diagonalized in terms of "massive" oscillators constructed out of the phase space fields in section 4.2. The re-normal-ordering constants produced in this process cancel between the bosons and the fermions and we precisely reproduce the spectrum obtained in the light-cone gauge together with the level-matching condition.

The final section, section 5, will be devoted to a discussion of some issues to be further clarified and future perspectives.

2. Classical analysis

2.1 The basic Lagrangian in the semi-light-cone gauge

The Lagrangian of the type IIB Green-Schwarz superstring in the plane-wave background with RR flux was constructed in [18, 50]. The basic fields are the string coordinates $X^\mu = (X^+, X^-, X^I)$ (where $X^\pm \equiv \frac{1}{\sqrt{2}}(X^9 \pm X^0)$ and $I = 1 \sim 8$) and the two sets of 16-component Majorana spinors $\theta^A_\alpha (A = 1, 2)$ of the same 10 dimensional chirality. We will essentially follow the convention of [18] with slight changes.[2] The 32×32 Γ^μ-matrices are decomposed

[2]In [18] Metsaev flipped the sign of X^μ just before writing down the form of the Lagrangian in the semi-light-cone gauge. We do not make this change and adopt his original convention.

into 16×16 γ^μ and $\bar\gamma^\mu$ matrices as $\Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \bar\gamma^\mu & 0 \end{pmatrix}$, where $(\gamma^\mu)^{\alpha\beta} = (1, \gamma^I, \gamma^9)^{\alpha\beta}$ and $(\bar\gamma^\mu)_{\alpha\beta} = (-1, \gamma^I, \gamma^9)_{\alpha\beta}$. A matrix Γ is defined as $\Gamma^\alpha{}_\beta \equiv (\gamma^1\bar\gamma^2\gamma^3\bar\gamma^4)^\alpha{}_\beta$ and it enjoys the property $\Gamma^2 = 1$. The worldsheet coordinates are denoted by $\xi^i = (t, \sigma), i = 0, 1$ and we define $\sigma_\pm \equiv t \pm \sigma$. The conventions for the flat worldsheet metric η_{ij} and the ϵ^{ij} tensor are $\eta_{ij} = (-1, +1)$ and $\epsilon^{01} = 1$.

After fixing the κ symmetry by imposing the semi-light-cone(SLC) gauge condition [47 – 49]

$$\bar\gamma^+\theta^A = 0\,, \tag{2.1}$$

the Lagrangian density is given by

$$\mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{WZ}}\,, \tag{2.2}$$

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &-\frac{T}{2}\sqrt{-g}g^{ij}\left(2\partial_i X^+\partial_j X^- - \mu^2 X_I^2 \partial_i X^+\partial_j X^+ + \partial_i X^I \partial_j X^I\right) \\ &+ iT\sqrt{-g}g^{ij}\left[\partial_i X^+(\theta^1\bar\gamma^-\partial_j\theta^1 + \theta^2\bar\gamma^-\partial_j\theta^2) + 2\mu\partial_i X^+\partial_j X^+\theta^1\bar\gamma^-\Gamma\theta^2\right]\,,\end{aligned} \tag{2.3}$$

$$\mathcal{L}_{\text{WZ}} = -\, iT\epsilon^{ij}\partial_i X^+(\theta^1\bar\gamma^-\partial_j\theta^1 - \theta^2\bar\gamma^-\partial_j\theta^2)\,. \tag{2.4}$$

$\mathcal{L}_{\text{kin}}$ is the kinetic part and $\mathcal{L}_{\text{WZ}}$ is the Wess-Zumino part. The quartic terms in $\mathcal{L}_{\text{kin}}$ proportional to μ^2 and μ describe, respectively, the coupling to the curved geometry and to the RR flux. We will keep the parameter μ carrying the dimension of mass throughout. We also keep the string scale explicitly. It is expressed through either the slope parameter α', the string tension $T = 1/2\pi\alpha'$, or the string length $\ell_s = \sqrt{\alpha'/2}$.

Due to the SLC gauge condition, we can reduce the fermions to 8-component SO(8) spinors and simplify the Lagrangian. In the basis where $\bar\gamma^9$ is diagonal, the spinor is decomposed as $\theta^A_\alpha = (\theta^A_a, \theta^A_{\dot a})$, and the SLC condition eliminates SO(8)-anti-chiral components $\theta^A_{\dot a}$. Furthermore, since the matrix Γ commutes with $\bar\gamma^9$, it can also be restricted to the SO(8)-chiral sector, retaining its basic properties $\Gamma_{ab}\Gamma_{bc} = \delta_{ac}$ and $\Gamma_{ab} = \Gamma_{ba}$. Hence, we may make a redefinition $\Gamma\theta^2 \to \theta^2$ and eliminate Γ altogether. The resultant Lagrangian reads

$$\begin{aligned}\mathcal{L}_{\text{kin}} = &-\frac{T}{2}\sqrt{-g}g^{ij}\left(2\partial_i X^+\partial_j X^- - \mu^2 X_I^2\partial_i X^+\partial_j X^+ + \partial_i X^I\partial_j X^I\right) \\ &+ i\sqrt{2}T\sqrt{-g}g^{ij}\left[\partial_i X^+(\theta^1\partial_j\theta^1 + \theta^2\partial_j\theta^2) + 2\mu\partial_i X^+\partial_j X^+\theta^1\theta^2\right]\,,\end{aligned} \tag{2.5}$$

$$\mathcal{L}_{\text{WZ}} = -\, i\sqrt{2}T\epsilon^{ij}\partial_i X^+(\theta^1\partial_j\theta^1 - \theta^2\partial_j\theta^2)\,, \tag{2.6}$$

where $\theta^1\partial_j\theta^1 \equiv \theta^1_a\partial_j\theta^1_a$, $\theta^1\theta^2 \equiv \theta^1_a\theta^2_a$, etc. This is the form to be used in the subsequent analysis.

2.2 Canonical analysis

The usual method for quantizing a field theory is to first find the complete set of solutions of the equations of motion and then set up the commutation relations among the time-independent coefficients in those solutions in such a way to realize the canonical equal-time commutation relations for the basic fields. In the case of string theory, one must , in addition, identify the Virasoro constraints and express them in terms of such quantized fields. In this subsection, we will describe what happens if one follows this canonical path. We will see that one encouters some unusual features for the system at hand.

2.2.1 Equations of motion and their solutions

Let us begin with the analysis of the equations of motion, with g_{ij} set equal to η_{ij}. The simplest is the equation for the field X^+. By varying $\mathcal{L}$ with respect to X^- one obtains $\partial_i \partial^i X^+ = 0$. So the solution is a free massless field and we will write it as

$$\mathcal{X}^+(\sigma_+, \sigma_-) = \mathcal{X}^+_L(\sigma_+) + \mathcal{X}^+_R(\sigma_-)\,, \tag{2.7}$$

$$\mathcal{X}^+_L(\sigma_+) = \frac{x^+}{2} + \ell_s^2 p^+ \sigma_+ + i\ell_s \sum_{n\neq 0} \frac{1}{n} \tilde{\alpha}^+_n e^{-in\sigma_+}\,, \tag{2.8}$$

$$\mathcal{X}^+_R(\sigma_-) = \frac{x^+}{2} + \ell_s^2 p^+ \sigma_- + i\ell_s \sum_{n\neq 0} \frac{1}{n} \alpha^+_n e^{-in\sigma_-}\,. \tag{2.9}$$

Here and hereafter, we use the calligraphic letters, such as $\mathcal{X}^+$, to denote the fields satisfying the equations of motion. The original notation, like X^+, refers to the one without such a requirement. This distinction will be very useful and important, especially in the sebsequent sections.

Next consider the equation of motion for X^I. Varying $\mathcal{L}$ with respect to X^I, we get

$$\partial_+\partial_- X^I + \mu^2(\partial_+\mathcal{X}^+_L \partial_- \mathcal{X}^+_R) X^I = 0\,. \tag{2.10}$$

The general solutions of this equation, 2π-periodic in σ, were given in the appendix of [46]. Let us quickly reproduce them. First make a change of variables and define the derivatives with respect to the new variables as follows:

$$(\sigma_+, \sigma_-) \to (\rho_+, \rho_-) \equiv (\mathcal{X}^+_L(\sigma_+), \mathcal{X}^+_R(\sigma_-))\,, \tag{2.11}$$

$$\tilde{\partial}_\pm \equiv \frac{\partial}{\partial \rho_\pm} = (\partial_\pm \rho_\pm)^{-1} \partial_\pm\,. \tag{2.12}$$

This produces the same effect as going to the light-cone gauge and the equation for X^I simplifies to

$$\tilde{\partial}_+ \tilde{\partial}_- X^I + \mu^2 X^I = 0\,. \tag{2.13}$$

Further form the following combinations:[3]

$$\tilde{t} \equiv \frac{1}{2\ell_s^2 p^+}(\rho_+ + \rho_-)\,, \qquad \tilde{\sigma} \equiv \frac{1}{2\ell_s^2 p^+}(\rho_+ - \rho_-)\,. \tag{2.14}$$

Under the shift $\sigma \to \sigma + 2\pi$, $\tilde{\sigma}$ undergoes the same shift $\tilde{\sigma} \to \tilde{\sigma} + 2\pi$, while $\tilde{t}$ is invariant. In terms of these variables, the equation (2.13) becomes

$$\left(\frac{\partial^2}{\partial \tilde{t}^2} - \frac{\partial^2}{\partial \tilde{\sigma}^2}\right) X^I + M^2 X^I = 0\,, \tag{2.15}$$

[3]Here and throughout, we take p^+ to be non-vanishing. In the Green-Schwarz formulation, such a restriction is necessary whenever we make an explicit separation of the first and the second class constraints.

where the dimensionless "mass" M is defined as

$$M \equiv \alpha' p^+ \mu = 2\ell_s^2 p^+ \mu \,. \tag{2.16}$$

This is nothing but the equation of motion for a free massive field and the general solution, 2π-periodic in σ, can be easily obtained. In terms of the original variables, it can be written in the following form:

$$\mathcal{X}^I = \sum_n (a_n^I u_n + \tilde{a}_n^I \tilde{u}_n) \,, \tag{2.17}$$

$$u_n = e^{-i(\omega_n \tilde{t} + n\tilde{\sigma})} = e^{-i(\lambda_n^+ \mathcal{X}_R^+ + \lambda_n^- \mathcal{X}_L^+)} \,, \tag{2.18}$$

$$\tilde{u}_n = e^{-i(\tilde{\omega}_n \tilde{t} - n\tilde{\sigma})} = e^{-i(\tilde{\lambda}_n^- \mathcal{X}_R^+ + \tilde{\lambda}_n^+ \mathcal{X}_L^+)} \,. \tag{2.19}$$

Here a_n^I and $\tilde{a}_n^I$ are constant coefficients and $\lambda_n^\pm$ etc. are given by

$$\lambda_n^\pm = \frac{1}{2\ell_s^2 p^+}(\omega_n \pm n) \,, \qquad \tilde{\lambda}_n^\pm = \frac{1}{2\ell_s^2 p^+}(\tilde{\omega}_n \pm n) \,, \tag{2.20}$$

$$\omega_n = \tilde{\omega}_n = \frac{n}{|n|}\sqrt{n^2 + M^2} \qquad \text{for } n \neq 0 \,, \tag{2.21}$$

$$\omega_0 = -\tilde{\omega}_0 = M \,. \tag{2.22}$$

Note that for $M \neq 0$ the functions u_n and $\tilde{u}_n$ inherently depend on both σ_- and σ_+ and hence it is hard to construct a purely right- or left-moving field out of $\mathcal{X}^I$.

The equations of motion for the fermions θ^A can be derived and solved in a similar way. In terms of the $\rho_\pm$ variables they become

$$\tilde{\partial}_+ \theta^1 = -\mu\theta^2 \,, \qquad \tilde{\partial}_- \theta^2 = \mu\theta^1 \,. \tag{2.23}$$

Combining them, θ^A satisfies the same equation as X^I (2.13), namely

$$\tilde{\partial}_+ \tilde{\partial}_- \theta^A + \mu^2 \theta^A = 0 \,. \tag{2.24}$$

Therefore the solution can be written in terms of the u_n and $\tilde{u}_n$ functions:

$$\vartheta^A = \sum_n (b_n^A u_n + \tilde{b}_n^A \tilde{u}_n) \,. \tag{2.25}$$

Putting this back into (2.23), the coefficients get related as

$$\mu b_n^2 = i\lambda_n^+ b_n^1 \,, \qquad \mu \tilde{b}_n^2 = i\tilde{\lambda}_n^- \tilde{b}_n^1 \,. \tag{2.26}$$

So only the half of these coefficients are independent.

Finally, let us consider the equation of motion for X^-, which is obtained by varying $\mathcal{L}$ with respect to X^+. After some simplification using the equations of motion for θ^A, it can be written in the $\rho_\pm$ coordinates as

$$\tilde{\partial}_+ \tilde{\partial}_- X^- = \mu^2 \mathcal{X}^I (\tilde{\partial}_+ + \tilde{\partial}_-) \mathcal{X}^I + i\sqrt{2}\mu(\vartheta^1 \tilde{\partial}_+ \vartheta^2 - \vartheta^2 \tilde{\partial}_- \vartheta^1) \,. \tag{2.27}$$

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The r.h.s. consists of already known functions and the l.h.s. is the Laplacian acting on X^-. Therefore X^- can be readily solved in terms of the other fields once we define a suitable inverse of the Laplacian. As we will not need the resultant rather complicated expression, we do not exhibit it. Obviously, we can always add a massless free field satisfying the homogeneous part of the equation, which is equivalent to $\partial_+\partial_-X^- = 0$. It is important to note that this free part can contain the modes of $\mathcal{X}^I$ and ϑ^A and can only be fixed by requiring the correct canonical commutation relations with all the other fields.

So we have obtained the general classical solutions of the system. They are expressed in terms of the basis functions u_n and $\tilde{u}_n$, which themselves depend on the modes of the field $\mathcal{X}^+$. Moreover, they depend both on σ_+ and σ_- inseparably, as already emphasized. We will now investigate how this situation is compatible with the existence of the right- and left-moving components of the energy-momentum tensor.

2.2.2 Energy-momentum tensor

Because the Lagrangian in the SLC gauge is classically conformally invariant, the ++ and the −− components of the energy-momentum tensor, to be denoted by $\mathcal{T}_\pm$, become the Virasoro constraints. Through a standard procedure one obtains

$$\begin{aligned}\frac{\mathcal{T}_\pm}{T} =& \frac{1}{2}\partial_\pm X^+\partial_\pm X^- + \frac{1}{4}(\partial_\pm X_I)^2 - \frac{i}{\sqrt{2}}\partial_\pm X^+(\theta^1\partial_\pm\theta^1 + \theta^2\partial_\pm\theta^2) \\ & - \frac{1}{4}(\partial_\pm X^+)^2(\mu^2X_I^2 + 4\sqrt{2}i\mu\theta^1\theta^2) \qquad (2.28)\\ =& (\partial_\pm\rho_\pm)^2\left[\frac{1}{2}\tilde{\partial}_\pm X^+\tilde{\partial}_\pm X^- + \frac{1}{4}(\tilde{\partial}_\pm X_I)^2 - \frac{i}{\sqrt{2}}\tilde{\partial}_\pm X^+(\theta^1\tilde{\partial}_\pm\theta^1 + \theta^2\tilde{\partial}_\pm\theta^2)\right. \\ & \left.- \frac{1}{4}(\tilde{\partial}_\pm X^+)^2(\mu^2X_I^2 + 4\sqrt{2}i\mu\theta^1\theta^2)\right], \qquad (2.29)\end{aligned}$$

where we have exhibited the form in the $\rho_\pm$ basis as well.

Now we substitute the equations of motion to see if $\mathcal{T}_\pm$ are functions of $\sigma_\pm$ respectively. Let us focus on $\mathcal{T}_+$. Using $\tilde{\partial}_+\mathcal{X}^+ = 1$ and $\mu\vartheta^2 = -\tilde{\partial}_+\vartheta^1$, it reduces to

$$\mathcal{T}_+ = \frac{T}{2}(\partial_+\rho_+)^2\left[\tilde{\partial}_+\mathcal{X}^- + \frac{1}{2}\left((\tilde{\partial}_+\mathcal{X}_I)^2 - \mu^2\mathcal{X}_I^2\right) - i\sqrt{2}(\vartheta^2\tilde{\partial}_+\vartheta^2 - \vartheta^1\tilde{\partial}_+\vartheta^1)\right]. \qquad (2.30)$$

Now let us act $\tilde{\partial}_-$ on the second and the third term in the square bracket. By using the equations of motion for $\mathcal{X}_I$ and ϑ^A, it is straightforward to get

$$\tilde{\partial}_-\left[\frac{1}{2}\left((\tilde{\partial}_+\mathcal{X}_I)^2 - \mu^2\mathcal{X}_I^2\right)\right] = -\mu^2\mathcal{X}^I(\tilde{\partial}_+ + \tilde{\partial}_-)\mathcal{X}^I\,, \qquad (2.31)$$

$$\tilde{\partial}_-\left[-i\sqrt{2}(\vartheta^2\tilde{\partial}_+\vartheta^2 - \vartheta^1\tilde{\partial}_+\vartheta^1)\right] = -i\sqrt{2}\mu(\vartheta^1\tilde{\partial}_+\vartheta^2 - \vartheta^2\tilde{\partial}_-\vartheta^1)\,. \qquad (2.32)$$

Note that the expressions on the r.h.s. are precisely those that appear in the equation of motion for X^- (2.27), with the signs reversed. This means that once-integrated equation of motion for X^- is

$$\tilde{\partial}_+\mathcal{X}^- = -\frac{1}{2}\left((\tilde{\partial}_+\mathcal{X}_I)^2 - \mu^2\mathcal{X}_I^2\right) + i\sqrt{2}(\vartheta^2\tilde{\partial}_+\vartheta^2 - \vartheta^1\tilde{\partial}_+\vartheta^1) + f_+(\sigma_+)\,, \qquad (2.33)$$

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where $f_+(\sigma_+)$ is an arbitrary function of σ_+. Substituting this into (2.30), $\mathcal{T}_+$ collapses to

$$\mathcal{T}_+ = \frac{T}{2}(\partial_+ \mathcal{X}_L^+)^2 f_+(\sigma_+)\,. \tag{2.34}$$

In an entirely similar manner, we get

$$\tilde{\partial}_- \mathcal{X}^- = -\frac{1}{2}\left((\tilde{\partial}_- \mathcal{X}_I)^2 - \mu^2 \mathcal{X}_I^2\right) - i\sqrt{2}(\vartheta^2 \tilde{\partial}_- \vartheta^2 - \vartheta^1 \tilde{\partial}_- \vartheta^1) + f_-(\sigma_-)\,, \tag{2.35}$$

$$\mathcal{T}_- = \frac{T}{2}(\partial_- \mathcal{X}_R^+)^2 f_-(\sigma_-)\,. \tag{2.36}$$

Thus we have a very unusual situation. Although $\mathcal{T}_\pm$ are indeed functions of $\sigma_\pm$ respectively, explicit dependence on the fields other than $\mathcal{X}^+$ is yet undetermined at this stage and hence invisible. Moreover, since $\mathcal{X}^I$ and ϑ^A consist of u_n and $\tilde{u}_n$ functions, it is not possible to construct a purely left-moving or right-moving field out of a local product of these fields. The only possible way for these fields to contribute to $f_\pm(\sigma_\pm)$ is through the coordinate-independent coefficients $a_n^I, \tilde{a}_n^I$, etc.

This is in striking contrast to the flat background case, for which μ is set to 0 right from the beginning. In that case, different fields are mutually independent and one obtains the familiar form of $\mathcal{T}_\pm$ for free massless fields. This shows that the $\mu \to 0$ limit is not smooth: No matter how small μ is, as long as it is finite the interactions connect up all the fields and lead to the unconventional result above.

This does not mean, however, that the system is inconsistent in this conformally invariant gauge. It only indicates that the conformal structure in a system where the left- and right-moving degrees of freedom are coupled is indeed quite subtle. What we need to do is to go to the next stage of the canonical analysis, namely to set up of the canonical Poisson-Dirac brackets for the fields, and try to find $f_\pm(\sigma_\pm)$ functions which realize the correct commutation relations among X^- and other fields.

2.2.3 Poisson-Dirac brackets for the fields and the modes

Let us now set up the Poisson Dirac bracket between the basic fields and their conjugates. We will denote the momenta conjugate to (X^+, X^-, X^I) as (P^-, P^+, P^I). They are given by

$$P^+ = T\partial_0 X^+\,, \tag{2.37}$$

$$\begin{aligned} P^- = T\big[&\partial_0 X^- - \partial_0 X^+(\mu^2 X_I^2 + 4\sqrt{2}i\mu\theta^1\theta^2) \\ &- 2\sqrt{2}i(\theta^1\partial_+\theta^1 + \theta^2\partial_+\theta^2)\big]\,, \end{aligned} \tag{2.38}$$

$$P^I = T\partial_0 X^I\,. \tag{2.39}$$

As for the fermionic fields, the momenta p^A conjugate to θ^A take the form

$$p^1 = i\sqrt{2}T(\partial_0 X^+ - \partial_1 X^+)\theta^1 = i\pi^{+1}\theta^1\,, \tag{2.40}$$

$$p^2 = i\sqrt{2}T(\partial_0 X^+ + \partial_1 X^+)\theta^2 = i\pi^{+2}\theta^2\,, \tag{2.41}$$

where

$$\pi^{+1} \equiv \sqrt{2}(P^+ - T\partial_1 X^+)\,, \qquad \pi^{+2} \equiv \sqrt{2}(P^+ + T\partial_1 X^+)\,. \tag{2.42}$$

Just as in the flat background case, these equations actually give the constraints

$$d^A \equiv p^A - i\pi^{+A}\theta^A = 0\,. \tag{2.43}$$

They simply say that p^A can be solved in terms of θ^A and hence they are of second class.

We define the Poisson brackets as

$$\left\{X^I(\sigma,t), P^J(\sigma',t)\right\}_P = \delta^{IJ}\delta(\sigma-\sigma')\,, \tag{2.44}$$

$$\left\{X^\pm(\sigma,t), P^\mp(\sigma',t)\right\}_P = \delta(\sigma-\sigma')\,, \tag{2.45}$$

$$\left\{\theta^A_a(\sigma,t), p^B_b(\sigma',t)\right\}_P = -\delta^{AB}\delta_{ab}\delta(\sigma-\sigma')\,, \tag{2.46}$$

$$\text{rest} = 0\,. \tag{2.47}$$

Under this bracket, the fermionic constraints d^A_a form the second class algebra

$$\left\{d^A_a(\sigma,t), d^B_b(\sigma',t)\right\}_P = 2i\delta^{AB}\delta_{ab}\pi^{+A}(\sigma,t)\delta(\sigma-\sigma')\,. \tag{2.48}$$

Defining the Dirac bracket in the standard way, θ^A's become self-conjugate and satisfy[4]

$$\left\{\theta^A_a(\sigma,t), \theta^B_b(\sigma',t)\right\}_D = \frac{i\delta^{AB}\delta_{ab}}{2\pi^{+A}(\sigma,t)}\delta(\sigma-\sigma')\,. \tag{2.49}$$

By going to the Dirac bracket, the relations (2.44) and (2.45) conitinue to hold, but the brackets between (X^-, P^-) and θ^A, which vanished under Poisson, become non-trivial.[5] One finds

$$\left\{X^-(\sigma,t), \theta^A(\sigma',t)\right\}_D = -\frac{1}{\sqrt{2}\pi^{+A}(\sigma,t)}\theta^A\delta(\sigma-\sigma')\,, \tag{2.50}$$

$$\left\{P^-(\sigma,t), \theta^A(\sigma',t)\right\}_D = -\frac{1}{\sqrt{2}\pi^{+A}(\sigma',t)}\theta^A(\sigma',t)\delta'(\sigma-\sigma')\,. \tag{2.51}$$

However, if we define the combination $\Theta^A_a \equiv \sqrt{2\pi^{+A}}\theta^A_a$, it is not difficult to check that they satisfy

$$\left\{\Theta^A_a(\sigma,t), \Theta^B_b(\sigma',t)\right\}_D = i\delta^{AB}\delta_{ab}\delta(\sigma-\sigma')\,, \tag{2.52}$$

and commute with all the other fields. So the fields to used are $(X^\pm, X^I, P^\pm, P^I, \Theta^A_a)$, which satisfy the canonical form of the Diract bracket relations.

We now come to the question of finding the brackets among the modes $a^I_n, \tilde{a}^I_n, b^A_n, \tilde{b}^A_n$, etc. which realize these canonical equal-time Dirac bracket relations for the fields. In the case of free fields, this is a textbook matter as one can easily express the modes in terms of the fields using the completeness relations of the basis functions $e^{in\sigma_\pm}$ at each time slice. Here we encounter a serious difficulty: Our basis functions, u_n and $\tilde{u}_n$ obtained in (2.18) and (2.19), are highly complicated functions of σ at equal t, since $\mathcal{X}^+_L$ and $\mathcal{X}^+_R$, appearing in the exponent, themselves depend on σ exponentially. Although we have been able to

[4]The inverse $1/\pi^{+A}$ is well-defined since its zero mode $\propto p^+$ is non-vanishing.

[5]The bracket $\left\{X^-, P^-\right\}_D$ still vanishes due to $\theta^A_a\theta^A_a = 0$.

express the modes, such as $a_n^I, \tilde{a}_n^I$, in terms of fields, but such expressions turned out to be quite formal and complicated, and so far not of practical use.

In this regard, note that if $\tilde{t}(t,\sigma)$ and $\tilde{\sigma}(t,\sigma)$, defined in (2.14), were the time and the space variables, the situation would have been much simpler, just like in the light-cone gauge. Thus, the difficulty we encountered is due to the non-trivial relation between the symplectic structures in the canonical (t,σ) basis and the $(\tilde{t},\tilde{\sigma})$ basis, which are connected by a field-dependent conformal transformation.

Fortunately, there is a nice way out of this problem. As we will explain in the next subsection, we can formulate the theory, including its dynamics, entirely in terms of the Fourier modes of the phase-space fields at $t=0$, without the use of the solutions of the equations of motion.

2.3 Phase-space formulation and the Virasoro algebra

2.3.1 Basic idea

In ordinary field theories, the knowledge of the Poisson(-Dirac) brackets of the phase-space fields at one time is not enough to describe the dynamics. Although one can promote these brackets into quantum brackets, it is not possible to compute the correlation functions of the fields at different times. This is why one needs to first obtain the solutions of the equations of motion and then find the brackets for the t-independent modes in them in order to construct the quantized fields at an arbitrary time.

The situation can be different, however, for a theory in which *the Hamiltonian is a member of the generators of a large symmetry algebra*. In such a case, provided that the symmetry is powerful enough, the representation theory of the algebra in the field space alone may fix the dynamics as well. String theories in a conformally invariant gauge belong to this category. To our knowledge, this observation has not been duly utilized in the past. This is simply because it has not been needed: Solvable string theories have been limited in number and the usual canonical procedure was sufficient to quantize and solve them. We would like to emphasize that the method to be described below is very powerful for the cases where the equations of motion are hard to solve due to non-linearity or, as in our case, the canonical quantization is difficult.

2.3.2 Classical Viraosoro algebra in the phase space

Hereafter we will be dealing with the phase-space fields, which are not subject to any equations of motion. The central objects are the energy-momentum tensors $\mathcal{T}_\pm$, which are given in (2.28). To express them in terms of the phase-space variables, it is convenient to introduce the following dimensionless fields $\{A, B, \widetilde{\Pi}, \Pi, S\}$, with appropriate sub- and super-scripts, and a dimensionless constant $\hat{\mu}$:

$$
X = \frac{1}{\sqrt{2\pi T}} A \,, \qquad\qquad P = \sqrt{\frac{T}{2\pi}} B \,,
$$

$$
\widetilde{\Pi} = \frac{1}{\sqrt{2}}(B + \partial_1 A) \,, \qquad\qquad \Pi = \frac{1}{\sqrt{2}}(B - \partial_1 A) \,,
$$

$$\Theta = -\frac{i}{\sqrt{2\pi}} S\,, \qquad\qquad \hat{\mu} = \frac{\mu}{\sqrt{2\pi T}}\,. \tag{2.53}$$

Then, $\mathcal{T}_\pm$ can be written as

$$\begin{aligned}
\mathcal{T}_+ &= \frac{1}{2}(\mathcal{H} + \mathcal{P}) \\
&= \frac{1}{2\pi}\left(\widetilde{\Pi}^+\widetilde{\Pi}^- + \frac{1}{2}\widetilde{\Pi}_I^2 + \frac{i}{2}S^2\partial_1 S^2 + \frac{\hat{\mu}^2}{2}\widetilde{\Pi}^+\Pi^+ A_I^2 - \frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+}S^1 S^2\right), \qquad (2.54)\\
\mathcal{T}_- &= \frac{1}{2}(\mathcal{H} - \mathcal{P}) \\
&= \frac{1}{2\pi}\left(\Pi^+\Pi^- + \frac{1}{2}\Pi_I^2 - \frac{i}{2}S^1\partial_1 S^1 + \frac{\hat{\mu}^2}{2}\widetilde{\Pi}^+\Pi^+ A_I^2 - \frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+}S^1 S^2\right). \qquad (2.55)
\end{aligned}$$

$\mathcal{P}$ and $\mathcal{H}$ are, respectively, the momentum density and the Hamiltonian density. Since the basic fields satisfy the canonical form of the Dirac bracket relations given in the previous subsection, we can compute the brackets among $\mathcal{P}$ and $\mathcal{H}$ and hence among $\mathcal{T}_\pm$. We need to make use of the following formulas for the derivatives of the δ-function for a general field $\mathcal{O}$:

$$\mathcal{O}(\sigma')\delta'(\sigma-\sigma') = \mathcal{O}(\sigma)\delta'(\sigma-\sigma') + \partial_1\mathcal{O}(\sigma)\delta(\sigma-\sigma')\,, \tag{2.56}$$

$$\mathcal{O}(\sigma')\delta''(\sigma-\sigma') = \mathcal{O}(\sigma)\delta''(\sigma-\sigma') + 2\partial_1\mathcal{O}(\sigma)\delta'(\sigma-\sigma') + \partial_1^2\mathcal{O}(\sigma)\delta(\sigma-\sigma')\,, \tag{2.57}$$

which must be understood in the sense of distributions. After a straightforward but long calculation, we obtain the expected results:

$$\begin{aligned}
\big\{\mathcal{H}(\sigma,t),\mathcal{H}(\sigma',t)\big\}_D &= \big\{\mathcal{P}(\sigma,t),\mathcal{P}(\sigma',t)\big\}_D \\
&= 2\mathcal{P}(\sigma,t)\delta'(\sigma-\sigma') + \partial_1\mathcal{P}(\sigma,t)\delta(\sigma-\sigma')\,, \qquad (2.58)\\
\big\{\mathcal{P}(\sigma,t),\mathcal{H}(\sigma',t)\big\}_D &= \big\{\mathcal{H}(\sigma,t),\mathcal{P}(\sigma',t)\big\}_D \\
&= 2\mathcal{H}(\sigma,t)\delta'(\sigma-\sigma') + \partial_1\mathcal{H}(\sigma,t)\delta(\sigma-\sigma')\,, \qquad (2.59)\\
\big\{\mathcal{T}_\pm(\sigma,t),\mathcal{T}_\pm(\sigma',t)\big\}_D &= \pm 2\mathcal{T}_\pm(\sigma,t)\delta'(\sigma-\sigma') \pm \partial_1\mathcal{T}_\pm(\sigma,t)\delta(\sigma-\sigma')\,, \qquad (2.60)\\
\big\{\mathcal{T}_\pm(\sigma,t),\mathcal{T}_\mp(\sigma',t)\big\}_D &= 0\,. \qquad (2.61)
\end{aligned}$$

The last two lines show that $\mathcal{T}_\pm$ form mutually commuting Virasoro algebras.[6] Integrating the first two equations with respect to σ' and identifying $\int d\sigma'\mathcal{H}(\sigma',t)$ to be the Hamiltonian H, which generates the time-development of a field $A(\sigma,t)$ as $\partial_0 A = \{A,H\}_D$, one readily finds

$$\partial_0\mathcal{H} = \{\mathcal{H},H\}_P = \partial_1\mathcal{P}\,, \qquad \partial_0\mathcal{P} = \{\mathcal{P},H\}_P = \partial_1\mathcal{H}\,. \tag{2.62}$$

Combining them, we get $\partial_\mp\mathcal{T}_\pm = 0$, showing that $\mathcal{T}_\pm = \mathcal{T}_\pm(\sigma_\pm)$. Therefore, we can define the Virasoro mode operators $T_n^\pm$ by

$$\mathcal{T}_\pm = \frac{1}{2\pi}\sum_n T_n^\pm e^{-in(t\pm\sigma)}\,. \tag{2.63}$$

[6] $\pm$ signs on the r.h.s. are just right for the Virasoro mode operators $T_n^\pm$ to satisfy the same algebra. See the discussion below.

Putting this into (2.60), one verifies that $T_n^{\pm}$ satisfy the usual form of the classical Virasoro algebra, namely

$$\left\{T_m^{\pm}, T_n^{\pm}\right\}_D = \frac{1}{i}(m-n)T_{m+n}^{\pm}\,, \qquad \left\{T_m^{\pm}, T_n^{\mp}\right\}_D = 0\,. \tag{2.64}$$

What is important here is that since $T_n^{\pm}$ are independent of t and σ they can be obtained from $\mathcal{T}_{\pm}$ at one time slice, which we take to be $t = 0$:

$$T_n^{\pm} = \int_0^{2\pi} d\sigma e^{\pm in\sigma}\mathcal{T}_{\pm}(\sigma, t=0)\,. \tag{2.65}$$

But at $t = 0$, we know the exact Dirac brackets for the fields composing $\mathcal{T}_{\pm}$ and hence we can quantize them in the standard way. Moreover, any properties of the system which are dictated by $T_n^{\pm}$ can in principle be calculable. The spectrum of physical states is one such quantity and in section 4 we will demonstrate that it can indeed be computed.

The vertex operators that create these physical states sould also be obtainable. Once they are constructed, one can calculate the correlation functions at unequal times, which reflect the dynamics of the system. This and the related matters will be investigated in a separate publication.

3. Quantization and quantum Virasoro operators

3.1 Quantization of basic fields

Let us now quantize the phase-space fields by replacing the Dirac brackets by the (anti-)commutators and $\delta(\sigma-\sigma') \to i\delta(\sigma-\sigma')$. As already said, we will do this at time $t = 0$ and hence drop t for all the fields from now on. The non-vanishing (anti-)commutators among the basic fields are then given by

$$\left[A^+(\sigma), B^-(\sigma')\right] = \left[A^-(\sigma), B^+(\sigma')\right] = 2\pi i\delta(\sigma-\sigma')\,, \tag{3.1}$$

$$\left[A^I(\sigma), B^J(\sigma')\right] = 2\pi i\delta^{IJ}\delta(\sigma-\sigma')\,, \tag{3.2}$$

$$\left\{S_a^A(\sigma), S_b^B(\sigma')\right\} = 2\pi\delta^{AB}\delta_{ab}\delta(\sigma-\sigma')\,. \tag{3.3}$$

We take the Fourier mode expansions of $A^\star, B^\star$ ($\star = (\pm, I)$) and S_a^A to be

$$A^\star(\sigma) = \sum_n A_n^\star e^{-in\sigma}\,, \qquad B^\star(\sigma) = \sum_n B_n^\star e^{-in\sigma}\,, \qquad S_a^A(\sigma) = \sum_n S_{a,n}^A e^{-in\sigma}\,. \tag{3.4}$$

Then, these modes satisfy the simple (anti-)commutation relations:

$$\left[A_m^{\pm}, B_n^{\mp}\right] = i\delta_{m+n,0}\,, \quad \left[A_m^I, B_n^J\right] = i\delta^{IJ}\delta_{m+n,0}\,, \tag{3.5}$$

$$\left\{S_{a,m}^A, S_{b,n}^B\right\} = \delta^{AB}\delta_{ab}\delta_{m+n,0}\,, \qquad \text{rest}=0\,. \tag{3.6}$$

As for the $\widetilde{\Pi}$ and Π fields, we have

$$\left[\widetilde{\Pi}^+(\sigma), \widetilde{\Pi}^-(\sigma')\right] = -\left[\Pi^+(\sigma), \Pi^-(\sigma')\right] = 2\pi i\delta'(\sigma-\sigma')\,,$$

$$\left[\widetilde{\Pi}^I(\sigma), \widetilde{\Pi}^J(\sigma')\right] = -\left[\Pi^I(\sigma), \Pi^J(\sigma')\right] = 2\pi i \delta^{IJ}\delta'(\sigma - \sigma'),$$
$$\left[\widetilde{\Pi}^I(\sigma), A^J(\sigma')\right] = \left[\Pi^I(\sigma), A^J(\sigma')\right] = -\frac{i}{\sqrt{2}}\delta^{IJ}\delta(\sigma - \sigma'). \tag{3.7}$$

From the definition, their modes are given by

$$\Pi_n^\pm \equiv \frac{1}{\sqrt{2}}(B_n^\pm + inA_n^\pm), \quad \Pi_n^I \equiv \frac{1}{\sqrt{2}}(B_n^I + inA_n^I), \tag{3.8}$$
$$\widetilde{\Pi}_n^\pm \equiv \frac{1}{\sqrt{2}}(B_n^\pm - inA_n^\pm), \quad \widetilde{\Pi}_n^I \equiv \frac{1}{\sqrt{2}}(B_n^I - inA_n^I), \tag{3.9}$$

and they satisfy the following commutation relations:

$$\left[\Pi_m^\pm, \Pi_n^\mp\right] = -m\delta_{m+n,0}, \qquad \left[\Pi_m^I, \Pi_n^J\right] = -m\delta^{IJ}\delta_{m+n,0}, \tag{3.10}$$
$$\left[\widetilde{\Pi}_m^\pm, \widetilde{\Pi}_n^\mp\right] = m\delta_{m+n,0}, \qquad \left[\widetilde{\Pi}_m^I, \widetilde{\Pi}_n^J\right] = m\delta^{IJ}\delta_{m+n,0}. \tag{3.11}$$

Note the difference in sign for Π commutators and $\widetilde{\Pi}$ commutators.

Let us make a remark. Although these quantum fields satisfy the free-field commutation relations, they are not bonafide free fields. The reason is that they time-develop non-trivially according to the Hamiltonian $H = \int d\sigma \mathcal{H}$, which contains non-quadratic terms. Nevertheless, one can compute all the commutators as if they were free massless fields. This is a great advantage of the present formalism.

3.2 Normal-ordering for quantum Virasoro operators

Now we want to define the quantum Virasoro operators with an appropriate normal-ordering and see if they form proper quantum Virasoro algebras. This turned out to be a rather delicate problem due precisely to the terms depending on the "mass" parameter $\hat{\mu}$.

To begin, let us recall the classical Virasoro algebra derived in (2.60). It tells us that both $-\mathcal{T}_-(\sigma)$ and $\mathcal{T}_-(-\sigma)$ satisfy the same algebra as $\mathcal{T}_+(\sigma)$ including the signs. In terms of modes, it means that $-T_{-n}^-$ and T_n^- satisfy the same standard form of the Virasoro algebra as T_n^+. Accordingly, we will study two different quantum extensions.

3.2.1 Phase-space normal-ordering

First, consider the case where we take the two sets of Virasoro operators to be

$$\mathcal{L}_+(\sigma) \equiv \mathcal{T}_+(\sigma) = \frac{1}{2\pi}\sum_n L_n^+ e^{-in\sigma}, \qquad L_n^+ \equiv T_n^+, \tag{3.12}$$
$$\mathcal{L}_-(\sigma) \equiv -\mathcal{T}_-(\sigma) = \frac{1}{2\pi}\sum_n L_n^- e^{-in\sigma}, \qquad L_n^- \equiv -T_{-n}^-. \tag{3.13}$$

The most natural normal-ordering scheme in this case is to regard $B_n^\star (n \geq 0), A_n^\star (n \geq 1), S_{a,n}^A (n \geq 1)$ as "annhiliation operators", where $\star = (\pm, I)$. We will call it the *phase-space normal-ordering*. Its naturalness can be seen, for instance, by quickly computing the central charge terms for the bosons in "$\pm$" sectors. As a matter of fact, since we know that

$\mathcal{L}_\pm$ form the standard Virasoro algebras classically, what we have to study are the quantum contributions, such as the central charge terms, coming from the "double-contractions".

Consider first the commutator $[\mathcal{L}_+(\sigma), \mathcal{L}_+(\sigma')]$, which is defined as $\mathcal{L}_+(\sigma - i\epsilon)\mathcal{L}_+(\sigma') - \mathcal{L}_+(\sigma' - i\epsilon)\mathcal{L}_+(\sigma)$, with an infinitesimal positive regulator ϵ. There are two different types of double contractions in this computation. One type consists of the usual contributions which are present for $\hat{\mu} = 0$ case as well. They produce the familiar c-number anomaly of the form $-(i/24\pi)(\delta'''(\sigma-\sigma') - \delta'(\sigma-\sigma'))$ for a boson and $(-i/24\pi)(\frac{1}{2}\delta'''(\sigma-\sigma') + \delta'(\sigma-\sigma'))$ for a self-conjugate periodic fermion. The other type, which will be our central focus from now on, consists of the contributions from the following commutators in the bosonic and the fermionic sectors respectively:

$$C_B = \frac{1}{(2\pi)^2}\left(\left[\frac{1}{2}\widetilde{\Pi}_I^2(\sigma), \frac{\hat{\mu}^2}{2}\widetilde{\Pi}^+\Pi^+ A_I^2(\sigma')\right] - (\sigma \leftrightarrow \sigma')\right), \tag{3.14}$$

$$C_F = \frac{1}{(2\pi)^2}\left[\frac{-i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+}S^1 S^2(\sigma), \frac{-i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+}S^1 S^2(\sigma')\right]. \tag{3.15}$$

As shown in appendix A.1, they yield the same non-vanishing singularity but with opposite signs, namely

$$C_B = -C_F = -\frac{i\hat{\mu}^2}{\pi}(2\widetilde{\Pi}^+\Pi^+\delta'(\sigma - \sigma') + \partial_\sigma(\widetilde{\Pi}^+\Pi^+)\delta(\sigma - \sigma')), \tag{3.16}$$

and hence cancel each other precisely. Similar cancellations take place also in $[\mathcal{L}_-(\sigma), \mathcal{L}_-(\sigma')]$ and $[\mathcal{L}_+(\sigma), \mathcal{L}_-(\sigma')]$. In this way we obtain the closed Virasoro algebra

$$\begin{aligned}\big[\mathcal{L}_\pm(\sigma), \mathcal{L}_\pm(\sigma')\big] = i\Big(&2\mathcal{L}_\pm(\sigma)\delta'(\sigma - \sigma') + \partial_\sigma\mathcal{L}_\pm(\sigma)\delta(\sigma - \sigma') \\ &- \frac{1}{24\pi}(14\delta'''(\sigma - \sigma') - 2\delta'(\sigma - \sigma'))\Big),\end{aligned} \tag{3.17}$$

$$\big[\mathcal{L}_+(\sigma), \mathcal{L}_-(\sigma')\big] = 0. \tag{3.18}$$

It should be emphasized that the bosonic string in the plane-wave background is conformally invariant classically but not quantum-mechanically: The contribution from the fermions coupled to the RR flux is crucial in cancelling the potential operator anomaly (3.16).

Now we have to discuss the issue of the central charge. As it stands, the central charge is only 14, 10 from the bosons and 4 from the self-conjugate fermions. We need to supply 12 more units to construct a consistent string theory. This situation, however, is exactly the same as for the flat background in the SLC gauge and the proper cure is known [51]. One only needs to add the quantum corrections of the form

$$\Delta\mathcal{L}_+ = -\frac{1}{2\pi}\partial_\sigma^2 \ln\widetilde{\Pi}^+, \qquad \Delta\mathcal{L}_- = \frac{1}{2\pi}\partial_\sigma^2 \ln\Pi^+, \tag{3.19}$$

to $\mathcal{L}_\pm$, respectively. It is straightforward to verify

$$\begin{aligned}&\big[\mathcal{L}_\pm(\sigma), \Delta\mathcal{L}_\pm(\sigma')\big] + \big[\Delta\mathcal{L}_\pm(\sigma), \mathcal{L}_\pm(\sigma')\big] \\ &\quad = i\Big(2\Delta\mathcal{L}_\pm(\sigma)\delta'(\sigma - \sigma') + \partial_\sigma(\Delta\mathcal{L}_\pm(\sigma))\delta(\sigma - \sigma') - \frac{1}{24\pi}12\delta'''(\sigma - \sigma')\Big).\end{aligned} \tag{3.20}$$

This means that $\Delta\mathcal{L}_\pm$ are primary operators of dimension 2, except that they provide the desired 12 units of central charge. Hereafter, $\mathcal{L}_\pm$ will be understood to include these quantum corrections.

3.2.2 Massless normal-ordering

Although we have already found that the phase-space normal-ordering scheme works, it is instructive to examine another scheme, one which appears natural for the case with $\hat{\mu} = 0$, i.e. for the usual free massless fields in 2 dimensions.

For $\hat{\mu} = 0$, the terms proportional to A_I^2 and S^1S^2 are absent and, in particular, the bosonic sector consists of $\widetilde{\Pi}^\star$ and $\Pi^\star$ fields only. This suggests that we may define the normal-ordering based on the modes of $\widetilde{\Pi}^\star$ and $\Pi^\star$. However, as we saw in (3.10), the commutators of $\Pi^\star_n$'s are opposite in sign to those of $\widetilde{\Pi}^\star_n$'s. Therefore, in the "$-$" sector, it is natural to reverse the mode number and define $\hat{\Pi}^\star_n \equiv \Pi^\star_{-n}$. At the field level, we introduce

$$\hat{\Pi}^\star(\sigma) \equiv \Pi^\star(-\sigma) = \sum_n \hat{\Pi}^\star_n e^{-in\sigma} \,. \tag{3.21}$$

Then, $\hat{\Pi}^\star$ statisfies the same commutation relation as $\widetilde{\Pi}^\star$, i.e. $\left[\hat{\Pi}^\star(\sigma), \hat{\Pi}^\star(\sigma')\right] = 2\pi i\delta'(\sigma - \sigma')$. Correspondingly, the Virasoro generator for the "$-$" sector should be taken as $\hat{\mathcal{T}}_-(\sigma) \equiv \mathcal{T}_-(-\sigma)$, which classically satisfies the standard form of the Virasoro algebra, as discussed at the beginning of section 3.2. The fermion S^1 appearing in the "$-$" sector should also be treated in a similar way. Namely we define $\hat{S}^1_n \equiv S^1_{-n}$ and introduce $\hat{S}^1(\sigma) \equiv S^1(-\sigma) = \sum_n \hat{S}^1_n e^{-in\sigma}$. What we shall call the *massless normal-ordering* is then defined by regarding $\widetilde{\Pi}^\star_n(n \geq 0)$, $\hat{\Pi}^\star_n(n \geq 0)$, $\hat{S}^1_n(n \geq 1)$, $S^2_n(n \geq 1)$ as "annihiliation operators" and will be denoted by the symbol ${}^{\times}_{\times}\quad{}^{\times}_{\times}$. For $\hat{\mu} = 0$, the Virasoro operators $\mathcal{T}_+(\sigma)$ and $\hat{\mathcal{T}}_-(\sigma)$ consist of mutually independent sets of fields and are readily shown to satisfy the isomorphic quantum Virasoro algebras of the standard form.

Let us try to extend this scheme to the $\hat{\mu} \neq 0$ case. We now have to express the non-zero modes of A^I in terms of $\widetilde{\Pi}^I_n$ and $\hat{\Pi}^I_n$ as

$$A^I_n = \frac{i}{\sqrt{2}}\frac{1}{n}(\widetilde{\Pi}^I_n - \hat{\Pi}^I_{-n}) \,. \tag{3.22}$$

The splitting of the field $A_I(\sigma)$ into the annihilation ((+)) and the creation(($-$)) parts should be made as

$$A_I(\sigma) = \mathcal{A}^{(+)}_I(\sigma) + \mathcal{A}^{(-)}_I(\sigma) \,, \tag{3.23}$$

$$\mathcal{A}^{(+)}_I(\sigma) = \frac{i}{\sqrt{2}}\sum_{n\geq 1}\frac{1}{n}\left(\widetilde{\Pi}_n e^{-in\sigma} + \hat{\Pi}_n e^{in\sigma}\right) \,, \tag{3.24}$$

$$\mathcal{A}^{(-)}_I(\sigma) = A_{I,0} - \frac{i}{\sqrt{2}}\sum_{n\geq 1}\frac{1}{n}\left(\widetilde{\Pi}_{-n} e^{in\sigma} + \hat{\Pi}_{-n} e^{-in\sigma}\right) \,. \tag{3.25}$$

Although A_I commutes with itself, $\mathcal{A}_I^{(+)}$ and $\mathcal{A}_I^{(-)}$ do not, and hence $\widetilde{\Pi}^+\Pi^+A_I^2$ must be defined with non-trivial normal-ordering, which discards the operator $\widetilde{\Pi}^+\Pi^+$ with a divergent coefficient.[7]

The Virasoro operators take the form (suppressing the normal-ordering symbol $\substack{\times\\\times}\ \ \substack{\times\\\times}$)

$$\begin{aligned}\mathcal{T}_+(\sigma) =& \frac{1}{2\pi}\bigg(\widetilde{\Pi}^+\widetilde{\Pi}^-(\sigma) + \frac{1}{2}\widetilde{\Pi}_I^2(\sigma) + \frac{i}{2}S^2\partial_1 S^2(\sigma) \\ &\qquad + \frac{\hat{\mu}^2}{2}\widetilde{\Pi}^+\Pi^+(\sigma)A_I^2(\sigma) - \frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+(\sigma)}\hat{S}^1(-\sigma)S^2(\sigma)\bigg)\,, \end{aligned} \tag{3.26}$$

$$\begin{aligned}\hat{\mathcal{T}}_-(\sigma) =& \frac{1}{2\pi}\bigg(\hat{\Pi}^+\hat{\Pi}^-(\sigma) + \frac{1}{2}\hat{\Pi}_I^2(\sigma) + \frac{i}{2}\hat{S}^1\partial_1\hat{S}^1(\sigma) \\ &\qquad + \frac{\hat{\mu}^2}{2}\hat{\tilde{\Pi}}^+\hat{\Pi}^+(\sigma)A_I^2(-\sigma) - \frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\hat{\tilde{\Pi}}^+\hat{\Pi}^+(\sigma)}\hat{S}^1(\sigma)S^2(-\sigma)\bigg)\,. \end{aligned} \tag{3.27}$$

An unusual feature is that due to the simultaneous presence of the hatted and unhatted operators, the mode number is not conserved for certain terms in the Virasoro generators. The calculation of the commutators between the Virasoro generators above proceeds in the similar way as before. Again the crucial part is the computation of the double-contraction contributions, which is described in appendix A.2.

There are several differences from the phase-space normal-ordering scheme. First, the contribution from the fermions, corresponding to (3.15), vanishes. On the other hand the bosonic contribution in the commutators $[\mathcal{T}_+(\sigma), \mathcal{T}_+(\sigma')]$ and $\left[\hat{\mathcal{T}}_-(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]$ is the same as C_B given in (3.16). Consequently, these Virasoro commutators contain extra uncanceled operator anomalies of the form

$$\left[\mathcal{T}_+(\sigma), \mathcal{T}_+(\sigma')\right]_{\text{extra}} = -\frac{i\hat{\mu}^2}{\pi}(2\widetilde{\Pi}^+\Pi^+(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma(\widetilde{\Pi}^+\Pi^+(\sigma))\delta(\sigma-\sigma'))\,, \tag{3.28}$$

$$\left[\hat{\mathcal{T}}_-(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]_{\text{extra}} = -\frac{i\hat{\mu}^2}{\pi}(2\hat{\tilde{\Pi}}^+\hat{\Pi}^+(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma(\hat{\tilde{\Pi}}^+\hat{\Pi}^+(\sigma))\delta(\sigma-\sigma'))\,. \tag{3.29}$$

Second, in the remaining commutator $\left[\mathcal{T}_+(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]$, a similar anomaly of the following form is produced from the bosonic sector:[8]

$$\left[\mathcal{T}_+(\sigma), \hat{\mathcal{T}}_-(\sigma')\right] = -\frac{i\hat{\mu}^2}{\pi}\partial_\sigma(\widetilde{\Pi}^+\Pi^+)(\sigma)\delta(\sigma+\sigma')\,. \tag{3.30}$$

As far as we know, these operator anomalies cannot be removed by adding some quantum correction to the Virasoro generators.

The conclusion is that while the massless normal-ordering works perfectly for $\hat{\mu}=0$ case, it is plagued with operator anomalies for $\hat{\mu}\neq 0$.

[7] Since, as can be easily checked, $\widetilde{\Pi}^+\Pi^+$ is an exactly marginal operator, subtraction of such an operator by itself does not interfere with conformal invariance.

[8] Due to the definition $\hat{\mathcal{T}}_-(\sigma) = \mathcal{T}_-(-\sigma)$, the argument of the δ-function below is $\sigma+\sigma'$, not $\sigma-\sigma'$. In any case, the point is that the Virasoro generators for the "±" sectors do not decouple.

4. Spectrum of physical states

4.1 BRST operator and its cohomology

With the quantum Virasoro generators defined with the phase-space normal-ordering, we can construct the nilpotent BRST operators Q and $\widetilde{Q}$ for the "$-$" and the "$+$" sectors in the usual way and study their cohomologies. As it will become evident, the analysis is exactly parallel to the case of the free bosonic string. This is because the structure of the unphysical quartets and the mechanism of their decoupling from the physical space is identical to that case, despite the presence of the additional interaction terms and (physical) fermions. Thus one may refer to the standard argument, a particularly suited one being that given in the Polchinski's book [52], and state the result. However, for self-containedness and for a need of some additional explanations, we shall recapitulate the essential part of the argument.

To make the presentation concise and transparent, it is convenient to first recall the following basic theorem on the cohomology, which will be repeatedly invoked.
Theorem: Let $(\mathcal{Q}, \mathcal{K}, \mathcal{N})$ be a triple of operators, $\mathcal{Q}, \mathcal{K}$ being fermionic and $\mathcal{N}$ bosonic, and assume that they satisfy the following relations:

$$\mathcal{Q}^2 = 0\,, \qquad \{\mathcal{Q}, \mathcal{K}\} = \mathcal{N}\,. \tag{4.1}$$

Then the cohomology of $\mathcal{Q}$ is in the kernel $\mathrm{Ker}\,(\mathcal{N})$ of $\mathcal{N}$.
Proof: The proof is simple. Let $|\Phi\rangle$ be a $\mathcal{Q}$-closed state, i.e. $\mathcal{Q}|\Phi\rangle = 0$. Then, from the relation above, $(\mathcal{Q}\mathcal{K} + \mathcal{K}\mathcal{Q})|\Phi\rangle = \mathcal{N}|\Phi\rangle = \mathcal{Q}\mathcal{K}|\Phi\rangle$. Now if $|\Phi\rangle$ is *not* in $\mathrm{Ker}\,(\mathcal{N})$, then, this can be rewritten as $|\Phi\rangle = \mathcal{Q}(\mathcal{N}^{-1}\mathcal{K}|\Phi\rangle)$, where we used the commutativity of $\mathcal{Q}$ and $\mathcal{N}$ which follows from (4.1). Thus, such a state is $\mathcal{Q}$-exact and hence the cohomology of $\mathcal{Q}$ can only be in $\mathrm{Ker}\,(\mathcal{N})$.

Now we begin the analysis for our system. Since the argument is entirely similar for Q and $\widetilde{Q}$, we will focus our attention on Q, which is given by

$$Q = \sum_n c_{-n} L_n^- - \frac{1}{2}\sum_{m,n} (m-n) :c_{-m} c_{-n} b_{m+n}: \,. \tag{4.2}$$

Define the "light-cone number" operator N_{lc} by

$$N_{lc} \equiv \sum_{n \geq 1} \frac{1}{n} \left(\Pi^-_{-n} \Pi^+_n - \Pi^+_{-n} \Pi^-_n \right)\,, \tag{4.3}$$

which assigns $+1$ to Π^+_n and -1 to Π^-_n for $n \neq 0$. Together with the non-zero modes of ghosts, $\Pi^\pm_n$ will form the unphysical quartet $q_n \equiv (\Pi^\pm_n, c_n, b_n)$. In terms of this grading, Q is split into

$$Q = Q_{-1} + Q_0 + Q_{\geq 1}\,, \tag{4.4}$$

where the subscript refers to the light-cone number. Q_{-1} is given by

$$Q_{-1} = -\hat{p}^+ \sum_{n \neq 0} \Pi^-_{-n} c_n\,, \qquad \hat{p}^+ \equiv \Pi^+_0\,, \tag{4.5}$$

while Q_0 is of the same form as Q in (4.2) except that $\Pi^\pm$ in L_n^- are replaced by their zero modes $\hat{p}^\pm$. The remaining piece $Q_{\geq 1}$ is complicated for our system but it contains at least one non-zero mode of $\Pi^\pm$. *The important point is that the explicit forms of Q_0 and $Q_{\geq 1}$ will not be required.* This is the reason why we can apply the reasoning for the free bosonic string to our case as well. The only information needed will be the relations that follow from the nilpotency $Q^2 = 0$ and the ghost number structure of the states.

Since $Q_{-1}^2 = 0$ follows immediately from $Q^2 = 0$ and the grading (4.4) above, one first considers the cohomology of this simple operator Q_{-1}. To make use of the basic theorem, introduce the operator

$$K \equiv \frac{1}{\hat{p}^+} \sum_{n \neq 0} \Pi^+_{-n} b_n \,. \tag{4.6}$$

Its anti-commutator with Q_{-1} produces a bosonic operator

$$N_q = \{K, Q_{-1}\} = \sum_{n=1}^{\infty} \left(n(b_{-n}c_n + c_{-n}b_n) - \Pi^-_{-n}\Pi^+_n - \Pi^+_{-n}\Pi^-_n \right) \,. \tag{4.7}$$

N_q counts the Virasoro level of the quartet members in the sense $[N_q, q_{-n}] = nq_{-n}$. Evidently, the set (Q_{-1}, K, N_q) forms a triple. Hence we can apply the basic theorem to learn that the cohomology of Q_{-1} must be in $\mathrm{Ker}\,(N_q)$, namely the transverse Hilbert space $\mathcal{H}_T$ where the quartet members are not excited.

In fact one can easily prove that the Q_{-1}- cohomology is equal to $\mathrm{Ker}\,(N_q)$. Although $\mathrm{Ker}\,(N_q)$ contains a sector with c_0 ghost, to make the presentation shorter, we will hereafter impose an additional condition $b_0|\Psi\rangle = 0$, as in [52], to eliminate such a sector.[9] In this setting, the states in $\mathrm{Ker}\,(N_q)$ carry the ghost number of the ghost vacuum, namely $-\frac{1}{2}$, and conversely any state with a ghost number different from this is not in $\mathrm{Ker}\,(N_q)$.

Now let us show first that all the states in $\mathrm{Ker}\,(N_q)$ are Q_{-1}-closed. Let $|\psi_0\rangle$ be in $\mathrm{Ker}\,(N_q)$, i.e. $N_q|\psi_0\rangle = 0$. Apply Q_{-1} and use the commutativity of Q_{-1} and N_q (which follows from (4.1)). Then, we get $N_q(Q_{-1}|\psi_0\rangle) = 0$. But since Q_{-1} carries ghost number 1, N_q on $Q_{-1}|\psi_0\rangle$ is non-vanishing. It follows that $Q_{-1}|\psi_0\rangle = 0$.

The proof that $|\psi_0\rangle$ is not Q_{-1}-exact is equally straightforward. Assume that $|\psi_0\rangle = Q_{-1}|\chi\rangle$ for some $|\chi\rangle$. Applying N_q one gets $0 = N_q|\psi_0\rangle = Q_{-1}(N_q|\chi\rangle)$, showing that $N_q|\chi\rangle$ is Q_{-1}-closed. But since $N_q|\chi\rangle$ is not in $\mathrm{Ker}\,(N_q)$ due to its ghost number, the basic theorem tells us that it must be of the Q_{-1}-exact form, i.e. $N_q|\chi\rangle = Q_{-1}|\xi\rangle$. As N_q is invertible in this sector, this means that $|\chi\rangle = Q_{-1}(N_q^{-1}|\xi\rangle)$. Hence $|\psi_0\rangle = Q_{-1}|\chi\rangle$ vanishes identically and there is no Q_{-1}-exact state in $\mathrm{Ker}\,(N_q)$.

One can apply exactly the same method to the study of Q-cohomology itself. To this end, one introduces another triple of operators (Q, K, N), where N is given by

$$N = \{Q, K\} \,. \tag{4.8}$$

[9] The full analysis including the c_0 sector can of course be done, with more involved ghost number analysis.

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Following the same logic as before, we immediately conclude that Q-cohomology is isomorphic to Ker (N).

The final step is to prove that in fact Ker (N) and Ker (N_q) are the same. This is done by an explicit construction of the isomophism between these spaces. For this purpose, we split the operator N into two parts as follows:

$$N = N_q + \check{N}, \,, \qquad \check{N} = \{Q_0 + Q_{\geq 1}, K\}\,. \tag{4.9}$$

Since K carries light-cone number 1, $\check{N}$ raises the light-cone number at least by 1 unit. In this sense, $\check{N}$ is an upper triangular matrix, while N_q is diagonal. Due to this structure, Ker (N) is no larger than Ker (N_q), i.e. Ker $(N_q) \supseteq$ Ker (N). To show the converse, let $|\psi_0\rangle$ be any member of Ker (N_q) and construct a state $|\Psi_0\rangle$ by

$$|\Psi_0\rangle = \frac{1}{1 + N_q^{-1}\check{N}}|\psi_0\rangle = (1 - N_q^{-1}\check{N} + (N_q^{-1}\check{N})^2 \cdots)|\psi_0\rangle\,. \tag{4.10}$$

N_q^{-1} is well-defined when operated after $\check{N}$, since $\check{N}$ raises the light-cone number at least by 1 unit. Operating $N = N_q + \check{N}$ from left, one readily finds $N|\Psi_0\rangle = N_q|\psi_0\rangle = 0$, showing the inclusion Ker $(N_q) \subseteq$ Ker (N).

We have now established the chain of isomorphisms: Q-cohomology $\simeq$ Ker $(N) \simeq$ Ker $(N_q) = \mathcal{H}_T \simeq Q_{-1}$-cohomology. Actually, there is one more condition on $\mathcal{H}_T$. It comes from the requirement $b_0|\Psi\rangle = 0$. Since both Q and b_0 annihilate the physical state $|\Psi\rangle$, we must have[10] $L_0^{-,tot}|\Psi\rangle = 0$, where $L_0^{-,tot} \equiv \{Q, b_0\}$. As $\mathcal{H}_T$ contains no ghost excitations, it is equivalent to the usual "on-shell condition" $L_0^-|\Psi\rangle = 0$.

Combining the result of an entirely analogous analysis for the "+" sector, we conclude that the physical states of the theory are the ones in $\mathcal{H}_T$ satisfying the conditions $L_0^\pm|\Psi\rangle = 0$.

Before concluding this subsection, we need to make a remark. In the preceding analysis, we have not been specific about the nature of the Hilbert space on which various operators act. According to our phase-space normal-ordering scheme, the natural space would be the Fock space $\mathcal{H}_{\rm Fock}$ built upon the oscillator vacuum $|0\rangle$ annihilated by the positive modes of $A_n^\star, B_n^\star, S_n^A$ and $B_0^\star$. However, as will be shown in the next subsection, non-trivial renormal-ordering will be required to diagonalize the on-shell conditions. This means that the physical eigenstates are not in $\mathcal{H}_{\rm Fock}$ and we must consider a larger Hilbert space as our arena. Precisely how large it should be is not clear at the moment and is left for future research. Nonetheless, as the cohomology analysis itself is fairly general and its essence is simply the decoupling of the unphysical quartet, its validity should be independent of such uncertainty.

4.2 Physical spectrum and comparison with the light-cone gauge formulation

Having shown that the spectrum of physical states is dictated by the $L_0^\pm$ constraints in the transverse Hilbert space $\mathcal{H}_T$ where the non-zero modes of $\Pi^\pm$ and $\widetilde{\Pi}^\pm$ (and all the ghosts) are removed, we now study these constraints in detail.

[10]Even in the treatment retaining the c_0 sector, the same condition arises: If $L_0^{-,tot}$ does not annihilate a state $|\Psi\rangle \in \mathcal{H}_T$, $|\Psi\rangle$ becomes Q-exact as $|\Psi\rangle = Q\left(\frac{1}{L_0^{-,tot}}b_0|\Psi\rangle\right)$, which is a contradiction.

Consider first the Hamiltonian constraint. Because of the redefinition (3.13), the (dimensionless) Hamiltonian should be identified as $H = L_0^+ - L_0^-$. In $\mathcal{H}_T$ it simplifies considerably and becomes quadratic in the modes. It takes the form (the phase-space normal-ordering is understood)

$$H = H_B + H_F \,, \tag{4.11}$$

$$H_B = \alpha' p^+ p^- + \frac{1}{2}\sum (B_{-n}^I B_n^I + (n^2 + M^2) A_{-n}^I A_n^I) \,, \tag{4.12}$$

$$H_F = \frac{1}{2}\sum(-n S_{-n}^1 S_n^1 + n S_{-n}^2 S_n^2 - iM S_{-n}^1 S_n^2 + iM S_{-n}^2 S_n^1) \,. \tag{4.13}$$

Obviously the bosonic part H_B describes the collection of free massive excitations and one can diagonalize it with ease. For the non-zero modes, we introduce the following oscillators:

$$\tilde{\alpha}_n^I = \frac{1}{\sqrt{2}}(B_n^I - i\omega_n A_n^I) \,, \qquad \alpha_n^I = \frac{1}{\sqrt{2}}(B_{-n}^I - i\omega_n A_{-n}^I) \,, \tag{4.14}$$

where ω_n is as defined in (2.21), i.e. $\omega_n = (n/|n|)\sqrt{n^2 + M^2}$ for $n \neq 0$. Using the commutation relations for A_n and B_n oscillators, we easily verify

$$[\tilde{\alpha}_m, \tilde{\alpha}_n] = [\alpha_m, \alpha_n] = \omega_n \delta_{m+n,0} \,, \qquad [\tilde{\alpha}_m, \alpha_n] = 0 \,. \tag{4.15}$$

Now we re-express the non-zero-mode part $H_B^{\neq 0}$ in terms of these oscillators, and re-normal-order such that $\alpha_n, \tilde{\alpha}_n$ for $n \geq 1$ are taken as annihiliation operators. This gives

$$H_B^{\neq 0} = \sum_{n\geq 1}(B_{-n}^I B_n^I + \omega_n^2 A_{-n}^I A_n^I) = \sum_{n\geq 1}(\alpha_{-n}^I \alpha_n^I + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I) + 8\sum_{n\geq 1}\omega_n \,, \tag{4.16}$$

where the last term is a divergent re-normal-ordering constant, suitably regularized. We shall see shortly that this gets canceled by the fermionic contribution. As for the zero mode part, we identify

$$\alpha_0^I = \frac{1}{\sqrt{2}}(B_0^I - iM A_0^I) \,. \tag{4.17}$$

Then, $\left[\alpha_0^I, {\alpha_0^J}^\dagger\right] = \delta^{IJ} M = \delta^{IJ}\omega_0$ and the zero mode part H_B^0 can be rewritten as

$$H_B^0 = \frac{1}{2}((B_0^I)^2 + M^2 (A_0^I)^2) = {\alpha_0^I}^\dagger \alpha_0^I + 4M \,. \tag{4.18}$$

In total H_B becomes

$$H_B = \alpha' p^+ p^- + {\alpha_0^I}^\dagger \alpha_0^I + \sum_{n\geq 1}(\alpha_{-n}^I \alpha_n^I + \tilde{\alpha}_{-n}^I \tilde{\alpha}_n^I) + 8\sum_{n\geq 1}\omega_n + 4M \,. \tag{4.19}$$

Next turn to H_F and again consider the non-zero mode part $H_F^{\neq 0}$ first. It can be written in the form

$$H_F^{\neq 0} = \sum_{n\geq 1}({S_n^1}^\dagger, {S_n^2}^\dagger) K(n) \begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} , \qquad K(n) = \begin{pmatrix} -n & -iM \\ iM & n \end{pmatrix} , \tag{4.20}$$

where $S_n^{A\dagger} = S_{-n}^A$. For each n, the hermitian matrix $K(n)$ is easily diagonalized by a unitary matrix $V(n)$ as

$$V(n)^\dagger K(n) V(n) = \begin{pmatrix} -\omega_n & 0 \\ 0 & \omega_n \end{pmatrix} . \tag{4.21}$$

Now we define a new basis of fermionic oscillators S_n and $\widetilde{S}_n$ by

$$\begin{pmatrix} S_n^\dagger \\ \widetilde{S}_n \end{pmatrix} = V^\dagger(n) \begin{pmatrix} S_n^1 \\ S_n^2 \end{pmatrix} . \tag{4.22}$$

The explicit expressions are

$$S_n = N(n)\left(S_{-n}^1 - i\frac{M}{\Omega_n^+}S_{-n}^2\right) , \qquad \widetilde{S}_n = N(n)\left(S_n^2 + i\frac{M}{\Omega_n^+}S_n^1\right) , \tag{4.23}$$

$$\Omega_n^+ \equiv \omega_n + n\,, \qquad N(n) \equiv \sqrt{\frac{\Omega_n^+}{\omega_n}} . \tag{4.24}$$

Obviously the new oscillators satisfy the same anti-commutation relations as the old ones, namely, $\left\{S_{a,m}, S_{b,n}^\dagger\right\} = \delta_{ab}\delta_{m,n}$, $\left\{\widetilde{S}_{a,m}, \widetilde{S}_{b,n}^\dagger\right\} = \delta_{ab}\delta_{m,n}$, and $\left\{S_{a,m}, \widetilde{S}_{b,n}^\dagger\right\} = 0$. Then we can rewrite $H_F^{\neq 0}$ as

$$\begin{aligned} H_F^{\neq 0} &= \sum_{n\geq 1}(S_n, \widetilde{S}_n^\dagger)\begin{pmatrix} -\omega_n & 0 \\ 0 & \omega_n \end{pmatrix}\begin{pmatrix} S_n^\dagger \\ \widetilde{S}_n \end{pmatrix} = \sum_{n\geq 1}\omega_n(-S_nS_n^\dagger + \widetilde{S}_n^\dagger\widetilde{S}_n) \\ &= \sum_{n\geq 1}\omega_n(S_n^\dagger S_n + \widetilde{S}_n^\dagger\widetilde{S}_n) - 8\sum_{n\geq 1}\omega_n\,, \end{aligned} \tag{4.25}$$

where in the last line we made a re-normal-ordering so that S_n for $n \geq 1$ are regarded as annihiliation operators. Notice that the re-normal-ordering constant generated by this process precisely cancels the one produced in the corresponding bosonic part, due to supersymmetry.

As for the zero mode part, the eigenvalues of the matrix $K(0)$ are $\pm M$ and defining the new zero mode by

$$S_0 \equiv \frac{1}{\sqrt{2}}(S_0^1 - iS_0^2)\,, \qquad \left\{S_0, S_0^\dagger\right\} = 1\,, \qquad (S_0)^2 = 0\,, \tag{4.26}$$

we can rewrite H_F^0 as

$$H_F^0 = -iMS_0^1S_0^2 = MS_0^\dagger S_0 - 4M\,. \tag{4.27}$$

Again the re-normal-ordering constant $-4M$ cancels the corresponding contribution in H_B^0.

Combining all the results, we find

$$\begin{aligned} H =& \alpha' p^+ p^- + \alpha_0^{I\dagger}\alpha_0^I + \sum_{n\geq 1}(\alpha_{-n}^I\alpha_n^I + \tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I) \\ &+ MS_0^\dagger S_0 + \sum_{n\geq 1}\omega_n(S_n^\dagger S_n + \widetilde{S}_n^\dagger\widetilde{S}_n)\,. \end{aligned} \tag{4.28}$$

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Setting this to zero and solving for $-p^-$, we precisely reproduce the familiar light-cone Hamiltonian H_{lc} computed in the light-cone gauge [18].

It remains to analyze the momentum constraint, which is expressed as $P = L_0^+ + L_0^- = 0$. In the transverse Hilbert space $\mathcal{H}_T$, P reduces to

$$P = \sum_{n\geq 1} in(A^I_{-n}B^I_n - B^I_{-n}A^I_n) + \sum_{n\geq 1}(nS^1_{-n}S^1_n + nS^2_{-n}S^2_n)\,. \tag{4.29}$$

As was done for the Hamiltonian, we rewrite it in terms of the new oscillators. Again the re-normal-ordering constants cancel between the bosonic and fermionic contributions and we find

$$P = \sum_{n\geq 1}\left(\frac{n}{\omega_n}\tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n + n\widetilde{S}^\dagger_n\widetilde{S}_n\right) - \sum_{n\geq 1}\left(\frac{n}{\omega_n}\alpha^I_{-n}\alpha^I_n + nS^\dagger_nS_n\right)\,. \tag{4.30}$$

Thus, as expected, $P = 0$ yields the level-matching condition.

5. Discussions

As already summarized in the introduction, we have been able to construct an exact world-sheet CFT description of the superstring in the plane-wave background with RR flux in terms of “free fields”. There are, however, several issues which require further understanding.

One is the relation of our formulation to the canonical approach, described in section 2.2, which unfortunately could not be pursued to the end due to a technical difficulty. This does not of course mean that the canonical approach should be abandoned. It would be very interesting if we can resurrect it by making use of the knowledge of the phase-space formulation.

Another point that should be clarified is the nature of the Hilbert space on which the Virasoro and the BRST operators act. As remarked in section 4.1, this has not yet been fully specified.

Let us now list some further future problems.

The most urgent is the construction of the primary operators, in particular the $(1,1)$ primaries corresponding to the low lying physical excitations. Our method developed in this paper gives priority to the quantization and the conformal symmetry structure and in a sense postpones the real dynamical issues. As already emphasized, the dynamical properties are encoded in the representation theory and by constructing the primary fields we can make them explicit. In this regard, one needs to understand as well the basic issue of what are the physical quantities to be computed in this background and how they should be compared to those in the super Yang-Mills theory.

Another obvious task is the understanding of the realization of the global symmetries of the theory [18, 50]. We should construct the generators of such symmetries in terms of the quantized fields and check that they close up to BRST- exact terms. This would shed further light on the justification of the normal-ordering we have adopted and the role of the quantum corrections required in the Virasoro generators.

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The modular invariance issue mentioned in the introduction can now be addressed in the proper setting. As shown in [35, 36], massive generalization of the elliptic functions appear and the further clarification of their properties would be interesting both physically and mathematically.

It would be an interesting project to use our CFT description as the starting point of a covariant pure spinor formalism in operator formulation. One way would be to apply the double-spinor formalism developed in [53], which allows one to derive the pure spinor superstring starting from a simple extension of the Green-Schwarz formalism.

Finally, it is a great challenge to try to apply the phase space formalism developed here to some suitable version of superstring theory in the $AdS_5 \times S^5$ background. In principle, one should be able to quantize the theory and construct the Virasoro operators just as we did for the plane-wave background, since the knowledge of the solutions of the equations of motion is not required. Of course the analysis of the spectrum would be much more difficult.

We hope to report on these and related matters in future communications.

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A. Double-contraction contributions in the Virasoro algebra with different normal-orderings

In this appendix, we display the computations of the double-contraction contributions in the Virasoro algebra in two different normal-ordering schemes and clarify their differences.

To facilitate the discussion, we first introduce a useful function that appears in the regularization of the commutators and prepare some formulas. Define the quantity q and a function $d(q, \epsilon)$ by

$$q \equiv e^{i(\sigma-\sigma')} , \tag{A.1}$$

$$d(q, \epsilon) \equiv \sum_{n \geq 0} \left(q e^{-\epsilon}\right)^n = \frac{1}{1 - q e^{-\epsilon}} , \tag{A.2}$$

where ϵ is an infinitesimal positive parameter. Then, it is straightforward to obtain the following formulas:

$$d(q, \epsilon) + d(q^{-1}, \epsilon) - 1 = 2\pi \delta_\epsilon(\sigma - \sigma') , \tag{A.3}$$

$$d(q, \epsilon) + d(q^{-1}, -\epsilon) = 1 , \tag{A.4}$$

$$d(q, \epsilon) - d(q, -\epsilon) = d(q^{-1}, \epsilon) - d(q^{-1}, -\epsilon) = 2\pi \delta_\epsilon(\sigma - \sigma') . \tag{A.5}$$

Here $\delta_\epsilon(\sigma-\sigma')$ is the usual regularized form of the δ-function given by

$$\delta_\epsilon(\sigma-\sigma') \equiv \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} e^{in(\sigma-\sigma')}e^{-|n|\epsilon} = \frac{1}{\pi}\frac{\epsilon}{(\sigma-\sigma')^2+\epsilon^2}\,. \tag{A.6}$$

Furthermore, the following formula involving the derivative of the δ-function will be useful:

$$\begin{aligned}2\pi\delta_\epsilon(\sigma-\sigma')\left[d(q^{-1},\epsilon)+d(q^{-1},-\epsilon)\right] &= d(q^{-1},\epsilon)^2-d(q^{-1},-\epsilon)^2\\ &= 2\pi i\delta'_\epsilon(\sigma-\sigma')+2\pi\delta_\epsilon(\sigma-\sigma')\,.\end{aligned} \tag{A.7}$$

To derive this relation, one must expand $d(q^{-1},\epsilon)$ up to the subleading order in powers of $\sigma-\sigma'-i\epsilon$.

A.1 Phase-space normal-ordering

Consider first the commutator $[\mathcal{T}_+(\sigma),\mathcal{T}_+(\sigma')]$ in the phase-space normal-ordering. In the bosonic sector, the double-contraction contributions of interest comes from

$$\frac{1}{(2\pi)^2}\left(\left[T(\sigma),V(\sigma')\right]+\left[V(\sigma),T(\sigma')\right]\right)\,, \tag{A.8}$$

where

$$T=\frac{1}{2}:\widetilde{\Pi}_I^2:\,,\qquad V=\chi:A_I^2:\,,\qquad \chi\equiv\frac{\hat{\mu}^2}{2}\widetilde{\Pi}^+\Pi^+\,. \tag{A.9}$$

Hereafter, we will drop the transverse subscript I for simplicity. The normal-ordering is defined by the splitting of fields $\widetilde{\Pi}=\widetilde{\Pi}^{(+)}+\widetilde{\Pi}^{(-)}$ and $A=A^{(+)}+A^{(-)}$, where the superscript $(+)$ ($(-)$) denotes the annihiliation (creation) part. In the phase-space normal-ordering, we have

$$\widetilde{\Pi}^{(+)}(\sigma)=\sum_{n\geq 0}\widetilde{\Pi}_n e^{-in\sigma}\,,\qquad \widetilde{\Pi}^{(-)}(\sigma)=\sum_{n\geq 1}\widetilde{\Pi}_{-n}e^{in\sigma}\,, \tag{A.10}$$

$$A^{(+)}(\sigma)=\sum_{n\geq 1}A_n e^{-in\sigma}\,,\qquad A^{(-)}(\sigma)=\sum_{n\geq 0}A_{-n}e^{in\sigma}\,. \tag{A.11}$$

The commutator $[T(\sigma),V(\sigma')]$ is defined by $T(\sigma-i\epsilon)V(\sigma')-V(\sigma'-i\epsilon)T(\sigma)$. It is easy to show that this amounts to taking the usual commutator once and then normal-order the remaining operator product. We thus obtain

$$\begin{aligned}\left[T(\sigma),V(\sigma')\right] &= -\frac{2\pi i}{\sqrt{2}}\chi(\sigma')\delta_\epsilon(\sigma-\sigma')(\widetilde{\Pi}(\sigma-i\epsilon)A(\sigma')+A(\sigma'-i\epsilon)\widetilde{\Pi}(\sigma))\\ &= -\frac{\pi i}{\sqrt{2}}\chi(\sigma')\delta_\epsilon(\sigma-\sigma'):\widetilde{\Pi}(\sigma)A(\sigma):\\ &\quad -\frac{2\pi i}{\sqrt{2}}\chi(\sigma')\delta_\epsilon(\sigma-\sigma')\left(\left[\widetilde{\Pi}^{(+)}(\sigma-i\epsilon),A^{(-)}(\sigma')\right]+\left[A^{(+)}(\sigma'-i\epsilon),\widetilde{\Pi}^{(-)}(\sigma)\right]\right)\,.\end{aligned}$$

The commutators in the second line are given by

$$\left[\widetilde{\Pi}^{(+)}(\sigma-i\epsilon),A^{(-)}(\sigma')\right]=-\frac{i}{\sqrt{2}}d(q^{-1},\epsilon)\,, \tag{A.12}$$

$$\left[A^{(+)}(\sigma'-i\epsilon),\widetilde{\Pi}^{(-)}(\sigma)\right]=-\frac{i}{\sqrt{2}}d(q^{-1},-\epsilon)\,. \tag{A.13}$$

We are interested in the double-contraction (DC) part given in the second line. Using the formula (A.7), we readily obtain

$$\big[T(\sigma), V(\sigma')\big]_{DC} = -\pi i\chi(\sigma')\delta'(\sigma-\sigma') - \pi\chi(\sigma)\delta(\sigma-\sigma')\,. \tag{A.14}$$

$[T(\sigma'), V(\sigma)]$ can be obtained from this by the interchange $\sigma \leftrightarrow \sigma'$.

Combining these results, the operator parts cancel and we get

$$\big[T(\sigma), V(\sigma')\big] + \big[V(\sigma), T(\sigma')\big] = -\pi i\left[2\chi(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma\chi(\sigma)\delta(\sigma-\sigma')\right]\,. \tag{A.15}$$

Now consider the contribution in the fermionic sector. The double-contraction contribution of interest comes from

$$\frac{1}{(2\pi)^2}\big[F(\sigma), F(\sigma')\big]\,, \tag{A.16}$$

where

$$F(\sigma) = \rho S^1(\sigma)S^2(\sigma)\,, \qquad \rho \equiv -\frac{i\hat{\mu}}{\sqrt{2}}\sqrt{\widetilde{\Pi}^+\Pi^+}\,. \tag{A.17}$$

Note the important relation $\rho^2 = -\chi$. The normal-ordering of the fermion S^A is defined by the splitting

$$S^A(\sigma) = S^{A(+)}(\sigma) + S^A_0 + S^{A(-)}(\sigma)\,, \tag{A.18}$$

$$S^{A(+)}(\sigma) = \sum_{n\geq 1} S^A_n e^{-in\sigma}\,, \qquad S^{A(-)}(\sigma) = \sum_{n\geq 1} S^A_{-n} e^{in\sigma}\,. \tag{A.19}$$

Defining the regularized commutator in the same way as in the case of the bosons, we get

$$\big[F(\sigma), F(\sigma')\big] = 2\pi\rho(\sigma)\rho(\sigma')\delta_\epsilon(\sigma-\sigma')\left(-S^2(\sigma-i\epsilon)S^2(\sigma') + S^1(\sigma'-i\epsilon)S^1(\sigma)\right)\,. \tag{A.20}$$

The products appearing here are normal-ordered as

$$S^2(\sigma-i\epsilon)S^2(\sigma') =: S^2(\sigma)S^2(\sigma') : + \frac{1}{2} + \left\{S^{2(+)}(\sigma-i\epsilon), S^{2(-)}(\sigma')\right\}, \tag{A.21}$$

$$S^1(\sigma'-i\epsilon)S^1(\sigma) =: S^1(\sigma')S^1(\sigma) : + \frac{1}{2} + \left\{S^{1(+)}(\sigma'-i\epsilon), S^{1(-)}(\sigma)\right\}, \tag{A.22}$$

where the term 1/2 is from the zero-mode and the anti-commutators are given by

$$\left\{S^{2(+)}(\sigma-i\epsilon), S^{2(-)}(\sigma')\right\} = d(q^{-1},\epsilon) - 1\,, \tag{A.23}$$

$$\left\{S^{1(+)}(\sigma'-i\epsilon), S^{1(-)}(\sigma)\right\} = d(q,\epsilon) - 1\,. \tag{A.24}$$

The normal-ordered products such as $: S^2(\sigma)S^2(\sigma') :$ vanish at $\sigma = \sigma'$ and hence do not contribute in the presence of $\delta_\epsilon(\sigma-\sigma')$ as above. Using the formulas for the $d(q,\epsilon)$ function and the relation $\rho^2 = -\chi$, we then get

$$\begin{aligned}\big[F(\sigma), F(\sigma')\big] &= 2\pi\rho(\sigma)^2\delta_\epsilon(\sigma-\sigma') - 2\pi\rho(\sigma)\rho(\sigma')\delta_\epsilon(\sigma-\sigma')(d(q^{-1},\epsilon) + d(q^{-1},-\epsilon))\\ &= \pi i\left(2\chi(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma\chi(\sigma)\delta(\sigma-\sigma')\right)\,.\end{aligned} \tag{A.25}$$

This precisely cancels the contribution (A.15) from the bosonic sector.

Computations for $[\mathcal{T}_-(\sigma), \mathcal{T}_-(\sigma')]$ and $[\mathcal{T}_+(\sigma), \mathcal{T}_-(\sigma')]$ are quite similar and the double-contractions again cancel between the bosonic and fermionic contributions. Hence, *the Virasoro algebra properly closes in the phase-space normal-ordering.*

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A.2 Massless normal-ordering

Again we begin with the bosonic sector of the commutator $[\mathcal{T}_+(\sigma), \mathcal{T}_+(\sigma')]$ and focus on the expression (A.8). The difference from the previous phase-space normal-ordering is that the field A must be split into the annihiliation and the creation parts in the following way:

$$A(\sigma) = \mathcal{A}^{(+)}(\sigma) + \mathcal{A}^{(-)}(\sigma)\,, \tag{A.26}$$

$$\mathcal{A}^{(+)}(\sigma) = \frac{i}{\sqrt{2}} \sum_{n\geq 1} \frac{1}{n}\left(\widetilde{\Pi}_n e^{-in\sigma} + \hat{\Pi}_n e^{in\sigma}\right)\,, \tag{A.27}$$

$$\mathcal{A}^{(-)}(\sigma) = A_0 - \frac{i}{\sqrt{2}} \sum_{n\geq 1} \frac{1}{n}\left(\widetilde{\Pi}_{-n} e^{in\sigma} + \hat{\Pi}_{-n} e^{-in\sigma}\right)\,. \tag{A.28}$$

The split for $\widetilde{\Pi}$ is as before, namely (A.10). Denoting this normal-ordering by $^{\times}_{\times}\quad{}^{\times}_{\times}$, $[T(\sigma), V(\sigma')]$ becomes

$$\begin{aligned}\big[T(\sigma), V(\sigma')\big] = &-\frac{\pi i}{\sqrt{2}}\chi(\sigma')\delta_\epsilon(\sigma-\sigma')\,{}^{\times}_{\times}\widetilde{\Pi}(\sigma)A(\sigma)\,{}^{\times}_{\times} \\ &- \frac{2\pi i}{\sqrt{2}}\chi(\sigma')\delta_\epsilon(\sigma-\sigma')\left(\Big[\widetilde{\Pi}^{(+)}(\sigma-i\epsilon), \mathcal{A}^{(-)}(\sigma')\Big] + \Big[\mathcal{A}^{(+)}(\sigma'-i\epsilon), \widetilde{\Pi}^{(-)}(\sigma)\Big]\right).\end{aligned}$$

The commutators in the last line are given by

$$\begin{aligned}\Big[\widetilde{\Pi}^{(+)}(\sigma-i\epsilon), \mathcal{A}^{(-)}(\sigma')\Big] &= -\frac{i}{\sqrt{2}}d(q^{-1},\epsilon)\,, \\ \Big[\mathcal{A}^{(+)}(\sigma'-i\epsilon), \widetilde{\Pi}^{(-)}(\sigma)\Big] &= -\frac{i}{\sqrt{2}}d(q^{-1},-\epsilon)\,,\end{aligned}$$

which are identical to (A.12) and (A.13). Therefore the rest of the calculations are also as before and we obtain

$$\big[T(\sigma), V(\sigma')\big] + \big[V(\sigma), T(\sigma')\big] = -\pi i\left[2\chi(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma\chi(\sigma)\delta(\sigma-\sigma')\right]\,. \tag{A.29}$$

This coincides with (A.15).

We now turn to the fermionic sector. Although the operator $F(\sigma)$ itself does not require normal-ordering, we must interpret $F(\sigma)$ as $F(\sigma) = \rho(\sigma)\hat{S}^1(-\sigma)S^2(\sigma)$, where $\hat{S}^1(-\sigma)$ is split into

$$\hat{S}^1(-\sigma) = \hat{S}^{1(+)}(-\sigma) + S^1_0 + \hat{S}^{1(-)}(-\sigma)\,, \tag{A.30}$$

$$\hat{S}^{1(+)}(-\sigma) = \sum_{n\geq 1}\hat{S}^1_n e^{in\sigma}\,, \tag{A.31}$$

$$\hat{S}^{1(-)}(-\sigma) = \sum_{n\geq 1}\hat{S}^1_{-n} e^{-in\sigma}\,. \tag{A.32}$$

The splitting for S^2 is the same as in the phase-space normal-ordering. Then the commutator $[F(\sigma), F(\sigma')]$ becomes

$$\big[F(\sigma), F(\sigma')\big] = 2\pi\rho(\sigma)\rho(\sigma')\delta_\epsilon(\sigma-\sigma')\left(-S^2(\sigma-i\epsilon)S^2(\sigma') + \hat{S}^1(-\sigma'-i\epsilon)\hat{S}^1(-\sigma)\right)\,. \tag{A.33}$$

Now $\hat{S}^1(-\sigma'-i\epsilon)\hat{S}^1(-\sigma)$ must be normal-ordered as

$$\hat{S}^1(-\sigma'-i\epsilon)\hat{S}^1(-\sigma) = {}^{\times}_{\times}\hat{S}^1(-\sigma')\hat{S}^1(-\sigma){}^{\times}_{\times} + \frac{1}{2} + \left\{\hat{S}^{1(+)}(-\sigma'-i\epsilon), \hat{S}^{1(-)}(-\sigma)\right\},$$

where the anticommutator is given by

$$\left\{\hat{S}^{1(+)}(-\sigma'-i\epsilon), \hat{S}^{1(-)}(-\sigma)\right\} = d(q^{-1},\epsilon) - 1\,. \tag{A.34}$$

The crucial difference from the phase-space normal-ordering is that, contrary to (A.24), $\left\{\hat{S}^{1(+)}(-\sigma'-i\epsilon), \hat{S}^{1(-)}(-\sigma)\right\}$ is identical to $\left\{S^{2(+)}(\sigma-i\epsilon), S^{2(-)}(\sigma')\right\}$ given in (A.23). Therefore the contributions from $\hat{S}^1$ and S^2 cancel in (A.33) and we get $[F(\sigma), F(\sigma')] = 0$.

Thus, in the massless normal-ordering scheme, the double-contraction contributions from the bosons and the fermions *do not* cancel and $[\mathcal{T}_+(\sigma), \mathcal{T}_+(\sigma')]$ contains an extra operator anomaly besides the usual c-number anomaly. Reinstating the factor of 8 (due to the transverse degrees of freedom), it reads

$$[\mathcal{T}_+(\sigma), \mathcal{T}_+(\sigma')]_{\text{extra}} = -\frac{2i}{\pi}(2\chi(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma\chi(\sigma)\delta(\sigma-\sigma'))\,. \tag{A.35}$$

The computation of $\left[\hat{\mathcal{T}}_-(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]$ is similar and the result again contains the operator anomaly:

$$\left[\hat{\mathcal{T}}_-(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]_{\text{extra}} = -\frac{2i}{\pi}(2\hat{\chi}(\sigma)\delta'(\sigma-\sigma') + \partial_\sigma\hat{\chi}(\sigma)\delta(\sigma-\sigma'))\,, \tag{A.36}$$

where $\hat{\chi}(\sigma) = \chi(-\sigma) = \hat{\bar{\Pi}}^+\hat{\Pi}^+$.

Finally, consider the commutator $\left[\mathcal{T}_+(\sigma), \hat{\mathcal{T}}_-(\sigma')\right]$. Actually, to make use of the various formulas already developed, it is more convenient to compute $\left[\mathcal{T}_+(\sigma), \hat{\mathcal{T}}_-(-\sigma')\right]$. For this quantity, while the fermionic contribution for the double-contraction part continues to vanish, a new situation occurs for the bosonic contribution. The relevant part is

$$\frac{1}{(2\pi)^2}\left(\left[T(\sigma), \hat{V}(-\sigma')\right] - \left[\hat{T}(-\sigma'), V(\sigma)\right]\right)\,, \tag{A.37}$$

The double-contraction part of $\left[T(\sigma), \hat{V}(-\sigma')\right]$ is easily seen to be the same as given in (A.14), namely

$$\left[T(\sigma), \hat{V}(-\sigma')\right]_{DC} = -\pi i\chi(\sigma')\delta'(\sigma-\sigma') - \pi\chi(\sigma)\delta(\sigma-\sigma')\,. \tag{A.38}$$

On the other hand, $\left[\hat{T}(-\sigma'), V(\sigma)\right]$ is not obtained simply by making the interchange $\sigma \leftrightarrow \sigma'$. Using the additional formulas

$$\begin{aligned}
\left[\hat{\Pi}^{(+)}(-\sigma'), \mathcal{A}^{(-)}(\sigma)\right] &= -\frac{i}{\sqrt{2}}d(q^{-1},\epsilon)\,,\\
\left[\mathcal{A}^{(+)}(\sigma), \hat{\Pi}^{(-)}(-\sigma')\right] &= -\frac{i}{\sqrt{2}}d(q^{-1},-\epsilon)\,,
\end{aligned}$$

we get

$$\left[\hat{T}(-\sigma'), V(\sigma)\right]_{DC} = -\pi i\chi(\sigma)\delta'(\sigma-\sigma') - \pi\chi(\sigma)\delta(\sigma-\sigma')\,. \tag{A.39}$$

Note that it is identical to $\left[T(\sigma), \hat{V}(-\sigma')\right]$ above, except for the argument of χ in front of $\delta'(\sigma-\sigma')$. Due to this difference, they do not quite cancel and produce an operator anomaly. Flipping the sign of σ' and supplying numerical factors, it is given by

$$\left[\mathcal{T}_{+}(\sigma), \hat{\mathcal{T}}_{-}(\sigma')\right] = -\frac{2i}{\pi}\partial_{\sigma}\chi(\sigma)\delta(\sigma+\sigma')\,. \tag{A.40}$$

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